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EXAMPLES OF SYSTEMS OF DIFFERENTIAL EQUATIONS...

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Examples of Systems of Differential Equations and Applications from Physics and the Technical Sciences

Calculus 4c-3

Examples of Systems of Differential Equations and Applications from Physics and the Technical Sciences – Calculus 4c-3

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Contents

Introduction

Here we present a collection of examples of general systems of linear differential equations and some applications in Physics and the Technical Sciences. The reader is also referred to *Calculus 4b* as well as to *Calculus 4c-2*.

It should no longer be necessary rigourously to use the ADIC-model, described in *Calculus 1c* and *Calculus 2c*, because we now assume that the reader can do this himself.

Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

Leif Mejlbro
21st May 2008

1 Homogeneous systems of linear differential equations

Example 1.1 *Given the homogeneous linear system of differential equations,*

$$(1)\quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad t \in \mathbb{R}.$$

1) Prove that everyone of the vectors

$$(2)\quad \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}, \qquad \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}, \qquad \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \qquad \begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix},$$

is a solution of (1).

2) Are the vectors in (2) linearly dependent or linearly independent?

3) How many linearly independent vectors can at most be chosen from (2)? In which ways can this be done?

4) Write down all solutions of (1).

5) Find that solution $\begin{pmatrix} x \\ y \end{pmatrix}$ *of (1), for which*

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

1) We shall just make a check:

$$\frac{d}{dt}\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} = \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} = \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix},$$

$$\frac{d}{dt}\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix},$$

$$\frac{d}{dt}\begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^1 \\ e^t \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix},$$

$$\frac{d}{dt}\begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix} = \begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix} = \begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix}.$$

2) The vectors are clearly linearly dependent, cf. also (3).

3) We can at most choose two linearly independent vectors. We have the following possibilities,

$$\left\{\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}, \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}, \begin{pmatrix} e^t \\ e^t \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}, \begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}, \begin{pmatrix} e^t \\ e^t \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}, \begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix}\right\}.$$

4) It follows from (3) that all solutions are e.g. given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} = \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_2 \cosh t + c_1 \sinh t \end{pmatrix},$$

for $t \in \mathbb{R}$, where c_1 and c_2 are arbitrary constants.

5) If we put $t = 0$ into the solution of (4), then

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

hence

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cosh t - \sinh t \\ -\cosh t + \sinh t \end{pmatrix} = \begin{pmatrix} e^{t} \\ -e^{-t} \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Example 1.2 *Prove that* $\begin{pmatrix} t+1 \\ t \end{pmatrix}$ *is a solution of the system*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1-t \\ -t \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Find all solutions of this system, and find in particular that solution, for which

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

If $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t+1 \\ t \end{pmatrix}$, then $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t+1 \\ t \end{pmatrix} + \begin{pmatrix} 1-t \\ -t \end{pmatrix} = \begin{pmatrix} t \\ t+1 \end{pmatrix} + \begin{pmatrix} 1-t \\ -t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix},$$

and the equation is fulfilled.

It follows from *Example 1.1* that the complete solution of the homogeneous system of equations is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}, \qquad c_1, c_2 \text{ arbitrære.}$$

Due to the linearity, the complete solution of the inhomogeneous system of differential equations is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t+1 \\ t \end{pmatrix} + c_1\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}, \quad c_1,\ c_2 \text{ arbitrære.}$$

If we put $t = 0$ into the complete solution, we get

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

hence $c_1 = 0$ and $c_2 = -1$. The wanted solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t+1 \\ t \end{pmatrix} - \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} = -\begin{pmatrix} t+1-\sinh t \\ t-\cosh t \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Example 1.3 *Find that solution* $\mathbf{z_1}(t) = (x_1, x_2)^T$ *of*

$$(3)\quad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

which satisfies $\mathbf{z_1}(0) = (1,0)^T$.
Than find that solution $\mathbf{z_2}(t)$ *of (3), which satisfies* $\mathbf{z_2}(0) = (0,1)^T$.
What is the complete solution of (3)?

1) The complete solution.

a) The **"fumbling method"**. The system is written

$$\begin{cases} dx_1/dt = x_1 - x_2, \\ dx_2/dt = x_1 + x_2, \end{cases} \qquad \text{thus in particular } x_2 = x_1 - \frac{dx_1}{dt}.$$

By insertion into the latter equation we get

$$\frac{dx_2}{dt} = \frac{dx_1}{dt} - \frac{d^2x_1}{dt^2} = x_1 + x_2 = x_1 + x_1 - \frac{dx_1}{dt},$$

hence by a rearrangement,

$$\frac{d^2x_1}{dt^2} - 2\,\frac{dx_1}{dt} + 2x_1 = 0.$$

The characteristic polynomial $R^2 - 2R + 2$ has the roots $R = 1 \pm i$, so we conclude that the complete solution is

$$x_1 = c_1e^t\cos t + c_2e^t\sin t, \qquad c_1, c_2 \text{ arbitrary}.$$

It follows from

$$\frac{dx_1}{dt} = (c_1 + c_2)e^t\cos t + (c_2 - c_1)e^t\sin t,$$

that

$$x_2 = x_1 - \frac{dx_1}{dt} = -c_2e^t\cos t + c_1e^t\sin t.$$

Summing up we get

$$(4)\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1e^t\cos t + c_2e^t\sin t \\ -c_2e^t\cos t + c_1e^t\sin t \end{pmatrix} = c_1e^t\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2e^t\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

b) **Alternatively** we apply the **eigenvalue method**. From

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 + 1 = 0$$

we obtain the complex conjugated eigenvalues $\lambda = 1 \pm i$.

A complex eigenvector for e.g. $\lambda = 1+i$ is the "cross vector" of $(1-\lambda,-1) = (-i,-1)$, thus e.g. $\mathbf{v} = (1,-i)$.
A *fundamental matrix* is

$$\mathbf{\Phi}(t) = \left(\mathrm{Re}\left\{e^{(a+i\omega)t}(\alpha+i\beta)\right\} \mid \mathrm{Im}\left\{e^{(a+i\omega)t}(\alpha+i\beta)\right\}\right) = e^{at}\cos\omega t(\alpha\,\beta) + e^{at}\sin\omega t(-\beta\,\alpha).$$

Here,

$$\lambda = 1+i = a+i\omega, \quad \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

so

$$\mathbf{\Phi}(t) = e^t\cos t\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + e^t\sin t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = e^t\begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}.$$

The complete solution is

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} = c_1e^t\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2e^t\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

c) **Alternatively** we can directly write down the **exponential matrix**,

$$\begin{aligned}\exp(\mathbf{A}t) &= e^{at}\left\{\cos\omega t - \frac{a}{\omega}\sin\omega t\right\}\mathbf{I} + \frac{1}{\omega}e^{at}\sin\omega t\cdot\mathbf{A} \\ &= e^t(\cos t - \sin t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^t\sin t\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = e^t\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},\end{aligned}$$

so the complete solution becomes

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{c} = c_1e^t\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2e^t\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

d) **Alternatively** (only sketchy) the eigenvalues $\lambda = 1 \pm i$ indicate that the solution necessarily is of the structure

$$\begin{cases} x_1(t) = a_1e^t\cos t + a_2e^t\sin t, \\ x_2(t) = b_1e^t\cos t + b_2e^t\sin t. \end{cases}$$

We have here four unknown constants, and we know that the final result may only contain two arbitrary constants. By insertion into the system of differential equations we get by an identification that $b_1 = a_1$ og $b_2 = -a_2$, and we find again the complete solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1e^t\cos t + a_2e^t\sin t \\ a_1e^t\sin t - a_2e^t\cos t \end{pmatrix} = a_1e^t\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + a_2e^t\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix},$$

where a_1 and a_2 are arbitrary constants.

2) By using the initial conditions $\mathbf{z_1}(0) = (1,0)^T$ in e.g. (4) we get

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

thus $c_1 = 1$ and $c_2 = 0$, and hence

$$\mathbf{z_1}(t) = \begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}.$$

3) By inserting the initial conditions $\mathbf{z_2}(0) = (0,1)^T$ into e.g. (4), we get

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

thus $c_1 = 0$ and $c_2 = -1$, hence

$$\mathbf{z_2}(t) = \begin{pmatrix} -e^t \sin t \\ e^t \cos t \end{pmatrix}.$$

4) The complete solution has already been given i four different versions in (1).

Example 1.4 *Find by using the eigenvalue method the complete solution of the following system of differential equations*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x}(t).$$

1) The **eigenvalue method**. It follows immediately that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$.
   To the eigenvalue $\lambda_1 = 1$ correspond the eigenvectors which are proportional to $(1,0)$.
   To the eigenvalue $\lambda_2 = -2$ corresponds the eigenvectors which are proportional to $(1,-3)$.
   The complete solution is
   $$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} e^t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} e^{-2t} \\ -3e^{-2t} \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix},$$
   where c_1 and c_2 are arbitrary constants.

2) **Alternatively** the **exponential matrix** is given by
   $$\begin{aligned} \exp(\mathbf{A}t) &= \frac{1}{\lambda_1 - \lambda_2}\left\{-\lambda_2 e^{\lambda_1 t} + \lambda_1 e^{\lambda_2 t}\right\} \mathbf{I} + \frac{1}{\lambda_1 - \lambda_2}\left\{e^{\lambda_1 t} - e^{\lambda_2 t}\right\} \mathbf{A} \\ &= \frac{1}{3}\left\{2e^t + e^{-2t}\right\} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3}\left\{e^t - e^{-2t}\right\} \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2e^t + e^{-2t} + e^t - e^{-2t} & e^t - e^{-2t} \\ 0 & 2e^t + e^{-2t} - 2e^t + 2e^{-2t} \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 3e^t & e^t - e^{-2t} \\ 0 & 3e^{-2t} \end{pmatrix}. \end{aligned}$$
   The complete solution is
   $$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} e^t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} e^t - e^{-2t} \\ 3e^{-2t} \end{pmatrix},$$
   where c_1 and c_2 are arbitrary constants.

3) **Alternatively** the system is written (the **"fumbling method"**),
   $$\frac{dx_1}{dt} = x_1 + x_2, \qquad \frac{dx_2}{dt} = -2x_2,$$
   from which we immediately get $x_2 = c_2 e^{-2t}$.
   Then by insertion
   $$\frac{dx_1}{dt} - x_1 = c_2 e^{-2t},$$
   so
   $$x_1 = c_1 e^t + c_2 e^t \int e^{-t} e^{-2t}\, dt = c_1 e^t - \frac{1}{3} c_2 e^{-2t}.$$
   Summing up we have
   $$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^t - \frac{1}{3} c_2 e^{-2t} \\ c_2 e^{-2t} \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{3} c_2 e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix},$$
   where c_1 and c_2 are arbitrary constants.

Example 1.5 *Find by the eigenvalue method the complete solution of the following system of differential equations*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix} \mathbf{x}(t).$$

1) The **eigenvalue method**. The eigenvalues are the solutions of the following equation,

$$\begin{vmatrix} 1-\lambda & 4 \\ -2 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda)+8 = \lambda^2+2\lambda+5=0,$$

hence $\lambda = -1 \pm 2i$.
A complex eigenvector corresponding to e.g.. $\lambda = a+i\omega = -1+2i$ is a cross vector of

$$(1-\lambda, 4) = (2-2i, 4) = 2(1-i, 2),$$

so we have e.g.

$$\mathbf{v} = \alpha + i\beta = (2, -1+i)^T = (2,-1)^T + i(0,1)^T.$$

Then a fundamental matrix is given by

$$\begin{aligned} \mathbf{\Phi}(t) &= e^{at}\cos\omega t(\alpha \;\; \beta) + e^{at}\sin\omega t(-\beta \;\; \alpha) \\ &= e^{-t}\cos 2t \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} + e^{-t}\sin 2t \begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} 2\cos 2t & 2\sin 2t \\ -\cos 2t - \sin 2t & \cos 2t - \sin 2t \end{pmatrix}. \end{aligned}$$

The complete solution is

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} 2\cos 2t \\ -\cos 2t - \sin 2t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}.$$

2) **Alternatively** it follows by the **"fumbling method"** that

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + 4x_2, \\[2mm] \dfrac{dx_2}{dt} = -2x_1 - 3x_2, \end{cases} \qquad \text{specielt } x_2 = \frac{1}{4}\,\frac{dx_1}{dt} - \frac{1}{4}\,x_1.$$

We get by insertion into the second equation,

$$\frac{1}{4}\,\frac{d^2x_1}{dt^2} - \frac{1}{4}\,\frac{dx_1}{dt} = -2x_1 - \frac{3}{4}\,\frac{dx_1}{dt} + \frac{3}{4}\,x_1,$$

hence by a rearrangement,

$$\frac{d^2x_1}{dt^t} + 2\,\frac{dx_1}{dt} + 5x_1 = 0.$$

The characteristic polynomial R^2+2R+5 has the roots $R=-1\pm 2i$, so the complete solution is

$$x_1(t)=c_1e^{-t}\cos 2t+c_2e^{-t}\sin 2t.$$

We conclude from

$$\frac{dx_1}{dt}=(2c_2-c_1)e^{-t}\cos 2t+(-2c_1-c_2)e^{-t}\sin 2t,$$

that

$$4x_2=\frac{dx_1}{dt}-x_1=(2c_2-2c_1)e^{-t}\cos 2t+(-2c_1-2c_2)e^{-t}\sin 2t.$$

Summing up we have

$$\begin{aligned}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} &= \begin{pmatrix}c_1e^{-t}\cos 2t+c_2e^{-t}\sin 2t\\ -\frac{1}{2}c_1e^{-t}(\cos 2t+\sin 2t)+\frac{1}{2}c_2e^{-t}(\cos 2t-\sin 2t)\end{pmatrix}\\ &= \frac{1}{2}c_1e^{-t}\begin{pmatrix}2\cos 2t\\-\cos 2t-\sin 2t\end{pmatrix}+\frac{1}{2}c_2e^{-t}\begin{pmatrix}2\sin 2t\\ \cos 2t-\sin 2t\end{pmatrix},\end{aligned}$$

where c_1 and c_2 are arbitrary constants.

3) **Alternatively** the **exponential matrix** is with $a=-1$ and $\omega=2$ given by

$$\begin{aligned}\exp(\mathbf{A}t) &= e^{at}\left\{\cos\omega t-\frac{a}{\omega}\sin\omega t\right\}\mathbf{I}+\frac{1}{\omega}e^{at}\sin\omega t\mathbf{A}\\ &= e^{-t}\left\{\cos 2t+\frac{1}{2}\sin 2t\right\}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{1}{2}e^{-t}\sin 2t\begin{pmatrix}1&4\\-2&-3\end{pmatrix}\\ &= e^{-t}\begin{pmatrix}\cos 2t+\sin 2t & 2\sin 2t\\ -\sin 2t & \cos 2t-\sin 2t\end{pmatrix},\end{aligned}$$

hence the complete solution is

$$\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}=c_1e^{-t}\begin{pmatrix}\cos 2t+\sin 2t\\-\sin 2t\end{pmatrix}+c_2e^{-t}\begin{pmatrix}2\sin 2t\\ \cos 2t-\sin 2t\end{pmatrix}.$$

4) **Alternatively** (sketch) the solution must have the following real structure,

$$\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}=\begin{pmatrix}a_1e^{-t}\cos 2t+a_2e^{-t}\sin 2t\\ b_1e^{-t}\cos 2t+b_2e^{-t}\sin 2t\end{pmatrix},$$

so we shall "only" check that this function satisfies the equations. The details are fairly long and tedious, so they are here left out.

Example 1.6 *Describe*

$$\begin{pmatrix} x''' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t^2 \\ t^3+1 \end{pmatrix}, \qquad t \in \mathbb{R},$$

as a linear system of differential equations of first order.

By introducing the new variables

$$x_1 = x, \qquad x_2 = x', \qquad x_3 = x'', \qquad x_4 = y,$$

the system can bow be written

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x''' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & -1 \\ 2 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ t^2 \\ t^3+1 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

There is here a very good reason for *not* asking about the complete solution. In fact, we see that the eigenvalues are the roots of the polynomial

$$\begin{aligned}\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 3 & 0 & -\lambda & -1 \\ 2 & 0 & 0 & 4-\lambda \end{vmatrix} &= -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & -1 \\ 0 & 0 & 4-\lambda \end{vmatrix} + 3 \begin{vmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ 0 & -\lambda & -1 \end{vmatrix} \\ &= -\lambda^3(4-\lambda)+3(4-\lambda)+2 = \lambda^4-4\lambda^3-3\lambda+14,\end{aligned}$$

where it can be proved that this polynomial does not have rationale roots.

Numerical calculations give approximatively

$$\lambda_1 = 1,56333, \quad \lambda_2 = 3,96633, \quad \left.\begin{matrix}\lambda_3 \\ \lambda_4\end{matrix}\right\} = -0,76483 \pm 1,29339i.$$

If one insists on solving the equation, the **"fumbling method"** is here without question the easiest one to apply. In fact, if we write the full system

$$\begin{cases} x''' = 3x - y + t^2, \\ y' = 2x + 4y + t^3 + 1, \end{cases} \quad \text{dvs. specielt } y = -x''' + 3x + t^2,$$

then it follows by insertion into the latter equation that

$$-x^{(4)} + 3x' + 2t = 2x - 4x^{(3)} + 12x + 4t^2 + t^3 + 1,$$

hence by a rearrangement

$$\frac{d^4x}{dt^4} - 4\frac{d^3x}{dt^3} - 3\frac{dx}{dt} + 14x = -t^3 - 4t^2 + 2t - 1.$$

The we guess a particular solution of the form of a polynomial of degree 3, at^3+bt^2+ct+d (the coefficients are really ugly), and since the characteristic polynomial is the same as before, we get the complete solution

$$x(t) = at^3 + bt^2 + ct + d + c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + c_3e^{\alpha t}\cos\beta t + c_4e^{\alpha t}\sin\beta t,$$

where we have $\lambda_3 = \alpha + i\beta$ and $\lambda_4 = \alpha - i\beta$ from above.
Then put this solution into

$$y = -x''' + 3x + t^2.$$

One has to admit that this method is somewhat easier to apply than the "standard method" of finding the eigenvectors first.

Example 1.7 *Find the complete solution of the system*

$$\frac{dx}{dt} = \frac{3}{2}x - y - \frac{1}{2}z,$$

$$\frac{dy}{dt} = -\frac{1}{2}x + 2y + \frac{1}{2}z,$$

$$\frac{dz}{dt} = \frac{1}{2}x + y = \frac{5}{2}z.$$

First solution. Inspection. It follows immediately that

$$\frac{d}{dt}(x+y) = x+y, \qquad \text{thus } x+y = 2a_1e^t,$$

$$\frac{d}{dt}(y+z) = 3(y+z), \qquad \text{thus } y+z = 2a_3e^{3t},$$

$$\frac{d}{dt}(z+x) = 2(z+x), \qquad \text{thus } z+x = 2a_2e^{2t},$$

so

$$\begin{cases} x = a_1e^t + a_2e^{2t} - a_3e^{3t}, \\ y = a_1e^t - a_2e^{2t} + a_3e^{3t}, \\ z = -a_1e^t + a_2e^{2t} + a_3e^{3t}, \end{cases}$$

or written as a vector,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a_1e^t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + a_2e^{2t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + a_3e^{3t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

where a_1, a_2 and a_3 are arbitrary constants.

Second solution. The eigenvalue method. The corresponding matrix

$$\mathbf{A} = \begin{pmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix}$$

has the characteristic polynomial

$$\begin{vmatrix} \frac{3}{2}-\lambda & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2}-\lambda \end{vmatrix} = (\frac{3}{2}-\lambda)(2-\lambda)(\frac{5}{2}-\lambda) - \frac{1}{4} + \frac{1}{4} + \frac{1}{4}(2-\lambda) - \frac{1}{2}(\frac{5}{2}-\lambda) - \frac{1}{2}(\frac{3}{2}-\lambda)$$

$$= -(\lambda-\frac{3}{2})(\lambda-2)(\lambda-\frac{5}{2}) - \frac{1}{4}(\lambda-2) + \frac{1}{2}(2\lambda-4) = (\lambda-2)\left\{-\left(\lambda-\frac{3}{2}\right)\left(\lambda-\frac{5}{2}\right) - \frac{1}{4} + 1\right\}$$

$$= -(\lambda-2)\left\{\lambda^2 - 4\lambda + \frac{15}{4} - \frac{3}{4}\right\} = -(\lambda-2)(\lambda^2-4\lambda+3) = -(\lambda-1)(\lambda-2)(\lambda-3),$$

so the eigenvalues are $\lambda = 1$, 2 and 3.

For $\lambda = 1$ we have the eigenvector $(1,1,-1)$.
For $\lambda = 2$ we have the eigenvector $(1,-1,1)$.
For $\lambda = 3$ we have the eigenvector $(-1,1,1)$.
The complete solution is then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

where c_1, c_2, c_3 are arbitrary constants.

Example 1.8 *Find the complete solution of the system*

$$\mathbf{Y}' = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{Y}.$$

The **eigenvalues** are the roots of the polynomial

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda-1)(\lambda+1)+2 = \lambda^2+1,$$

thus $\lambda = \pm i$. Since the eigenvalues are complex numbers, we have four solution variants.

1) The **eigenvalue method**. To $\lambda = a + i\omega = i$, i.e. $a = 0$ and $\omega = 1$, we have a complex eigenvector of the form

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \alpha + i\beta.$$

Then a fundamental matrix is given by

$$\begin{aligned} \mathbf{\Phi}(t) &= \left(\operatorname{Re}\left\{e^{(a+i\omega)t}(\alpha+i\beta)\right\} \;\; \operatorname{Im}\left\{e^{(a+i\omega)t}(\alpha+i\beta)\right\}\right) = e^{at}\cos\omega t(\alpha \;\; \beta) + e^{at}\sin\omega t(-\beta \;\; \alpha) \\ &= \cos t \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} + \sin t \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ \cos t + \sin t & \sin t - \cos t \end{pmatrix}, \end{aligned}$$

so the complete solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}, \qquad c_1, c_2 \text{ arbitrary.}$$

2) The **exponential matrix**. Since the eigenvalues are complex conjugated, the exponential matrix is given by a formula ($a = 0$ and $\omega = 1$),

$$\begin{aligned} \exp(\mathbf{A}t) &= e^{at}\left\{\cos\omega t - \frac{a}{\omega}\sin\omega t\right\}\mathbf{I} + \frac{1}{\omega}e^{at}\sin\omega t \cdot \mathbf{A} = \cos t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin t \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \cos t + \sin t & -\sin t \\ 2\sin t & \cos t - \sin t \end{pmatrix}. \end{aligned}$$

Then the complete solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} \cos t + \sin t \\ 2\sin t \end{pmatrix} + c_2 \begin{pmatrix} -\sin t \\ \cos t - \sin t \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

3) Since $\lambda = \pm i$, the **real structure** of the solution is given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_1 \cos t + a_2 \sin t \\ b_1 \cos t + b_2 \sin t \end{pmatrix},$$

hence

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_2 \cos t - a_1 \sin t \\ b_2 \cos t - b_1 \sin t \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a_1 \cos t + a_2 \sin t \\ b_1 \cos t + b_2 \sin t \end{pmatrix} = \begin{pmatrix} (a_1-b_1)\cos t+(a_2-b_2)\sin t \\ (2a_1-b_1)\cos t+(2a_2-b_2)\sin t \end{pmatrix}.$$

When we identify the coefficients, we eliminate b_1 and b_2, thus

$$a_2 = a_1 - b_1 \qquad \text{and} \qquad -a_1 = a_2 - b_2,$$

and hence

$$b_1 = a_1 - a_2 \qquad \text{and} \qquad b_2 = a_1 + a_2.$$

The complete solution is then

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_1 \cos t + a_2 \sin t \\ (a_1 - a_2)\cos t + (a_1 + a_2)\sin t \end{pmatrix} = a_1 \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + a_2 \begin{pmatrix} \sin t \\ -\cos t + \sin t \end{pmatrix},$$

where a_1 and a_2 are arbitrary constants.

4) The **"fumbling method"**. It follows from

$$\frac{dy_1}{dt} = y_1 - y_2, \qquad \text{i.e.} \qquad y_2 = -\frac{dy_1}{dt} + y_1,$$

$$\frac{dy_2}{dt} = 2y_1 - y_2,$$

by eliminating y_2 that

$$-\frac{d^2y_1}{dt^2} + \frac{dy_1}{dt} = 2y_1 + \frac{dy_1}{dt} - y_1,$$

hence by a rearrangement

$$\frac{d^2y_1}{dt^2} + y_1 = 0.$$

Then we get the complete solution

$$y_1 = c_1 \cos t + c_2 \sin t.$$

This gives us

$$\begin{aligned} y_2 &= -\frac{dy_1}{dt} + y_1 = -(-c_1 \sin t + c_2 \cos t) + c_1 \cos t + c_2 \sin t \\ &= c_1(\sin t + \cos t) + c_2(-\cos t + \sin t). \end{aligned}$$

Summing up the complete solution becomes

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} \cos t \\ \sin t + \cos t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t + \sin t \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

Example 1.9 *Find a fundamental matrix of the system*

$$y_1' = 2y_1 + 5y_2 - 3y_3,$$
$$y_2' = -y_1 - 2y_2 + y_3,$$
$$y_3' = y_1 + y_2.$$

The equation is written in matrix form

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 & 5 & -3 \\ -1 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

The **eigenvalues** are the roots of the polynomial

$$\begin{vmatrix} 2-\lambda & 5 & -3 \\ -1 & -2-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 5 & -3 \\ \lambda & -2-\lambda & 1 \\ \lambda & 1 & -\lambda \end{vmatrix} = \lambda\begin{vmatrix} -1 & 5 & -3 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$$
$$= \lambda\begin{vmatrix} -1 & 5 & -3 \\ 0 & 3-\lambda & -2 \\ 0 & 6 & -3-\lambda \end{vmatrix} = -\lambda\begin{vmatrix} 3-\lambda & -2 \\ 6 & -3-\lambda \end{vmatrix} = -\lambda(\lambda^2-9+12) = -\lambda(\lambda^2+3),$$

thus the eigenvalues are $\lambda = 0$ and $\lambda = \pm i\sqrt{3}$.

An eigenvector (a_1, b_1, c_1) corresponding to $\lambda = 0$ satisfies

$$\left\{\begin{array}{r} 2a_1 + 5b_1 - 3c_1 = 0, \\ -a_1 - 2b_1 + c_1 = 0, \\ a_1 + b_1 = 0, \end{array}\right. \qquad \text{dvs.} \quad \left\{\begin{array}{l} b_1 = -a_1, \\ c_1 = a_1 + 2b_1 = -a_1. \end{array}\right.$$

Hence we may e.g. choose $(1, -1, -1)$.

An eigenvector (a_2, b_2, c_2) corresponding to $\lambda = i\sqrt{3}$ satisfies

$$\left\{\begin{array}{r} 2a_2+5b_2-3c_2 = i\sqrt{3}a_2, \\ -a_2-2b_2+c_2 = i\sqrt{3}b_2, \\ a_2+b_2 = i\sqrt{3}c_2, \end{array}\right. \quad \text{dvs.} \quad \left\{\begin{array}{l} 5b_2-3c_2 = (-2+i\sqrt{3})a_2, \\ (-2-i\sqrt{3})b_2+c_2 = a_2, \\ b_2-i\sqrt{3}c_2 = -a_2. \end{array}\right.$$

It follows from the latter two equations by an addition

$$-(1+i\sqrt{3})b_2 + (1-i\sqrt{3})c_2 = 0,$$

hence

$$c_2 = \frac{1+i\sqrt{3}}{1-i\sqrt{3}}b_2 = \frac{(1+i\sqrt{3})^2}{1+3}\,b_2 = \frac{1}{4}\cdot(1-3+2i\sqrt{3})b_2 = \frac{1}{2}(-1+i\sqrt{3})b_2.$$

By insertion into the second equation we get

$$a_2 = (-2-i\sqrt{3})b_2 + \frac{1}{2}(-1+i\sqrt{3})b_2 = \frac{1}{2}(-5-i\sqrt{3})b_2.$$

By choosing $b_2 = 2$ we find the eigenvector

$$(-5-i\sqrt{3}, 2, -1+i\sqrt{3})^T.$$

We get by a complex conjugation that an eigenvector corresponding to $\lambda = -i\sqrt{3}$ is given by

$$(-5+i\sqrt{3}, 2, -1-i\sqrt{3})^T.$$

The latter two columns of the corresponding fundamental matrix are

$$\cos\sqrt{3}t(\alpha \quad \beta) + \sin\sqrt{3}t(-\beta \quad \alpha) = \cos(\sqrt{3}t)\begin{pmatrix} -5 & -\sqrt{3} \\ 2 & 0 \\ -1 & \sqrt{3} \end{pmatrix} + \sin(\sqrt{3}t)\begin{pmatrix} \sqrt{3} & -5 \\ 0 & 2 \\ -\sqrt{3} & -1 \end{pmatrix},$$

hence a fundamental matrix is given by

$$\mathbf{\Phi}(t) = \begin{pmatrix} 1 & -5\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t & -\sqrt{3}\cos\sqrt{3}t - 5\sin\sqrt{3}t \\ -1 & 2\cos\sqrt{3}t & 2\sin\sqrt{3}t \\ -1 & -\cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t & \sqrt{3}\cos\sqrt{3}t - \sin\sqrt{3}t \end{pmatrix}.$$

Example 1.10 *Find the complete solution of the system*

$$\mathbf{Y}' = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\mathbf{Y}.$$

Obviously, $\lambda = 1$ is an **eigenvalue** of multiplicity 2. We have a couple of solution methods.

1) **Discussion of the structure of the solution**. The algebraic multiplicity is 2, while the geometric multiplicity is only w. Hence the complete solution must necessarily have the structure

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_1e^t + a_2te^t \\ b_1e^t + b_2te^t \end{pmatrix}.$$

It follows by a couple of calculations that

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (a_1+a_2)e^t + a_2te^t \\ (b_1+b_2)e^t + b_2te^t \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a_1e^t + a_2te^t \\ b_1e^t + b_2te^t \end{pmatrix} = \begin{pmatrix} a_1e^t + a_2te^t \\ (2a_1+b_1)e^t + (2a_2+b_2)te^t \end{pmatrix}.$$

When we identify the coefficients we find that

$$\begin{aligned} a_1 + a_2 &= a_1, && \text{thus } a_2 = 0, \\ b_1 + b_2 &= 2a_1 + b_1, && \text{thus } b_2 = 2a_1. \end{aligned}$$

The two free parameters are a_1 and b_1, while $a_2 = 0$ and $b_2 = 2a_1$, so

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_1e^t \\ b_1e^t + 2a_1te^t \end{pmatrix} = a_1e^t\begin{pmatrix} 1 \\ 2t \end{pmatrix} + b_1e^t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 2te^t & e^t \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix},$$

where a_1 and b_1 are arbitrary constants.

2) The **exponential matrix**. Since $\mathbf{A}$ and $\mathbf{I}$ commute, the exponential matrix is given by

$$\exp(\mathbf{A}t) = \exp((\mathbf{A}-\mathbf{I})t + \mathbf{I}t) = e^t \exp(\mathbf{B}t),$$

where

$$\mathbf{B} = \mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

and where $\mathbf{B}^2 = \mathbf{0}$, thus $\mathbf{B}^n = \mathbf{0}$ for $n \geq 2$. Then

$$\begin{aligned} \exp(\mathbf{A}t) &= e^t \exp(\mathbf{B}t) = e^t \left\{ \mathbf{I} + \mathbf{B}t + \sum_{n=2}^{\infty} \frac{1}{n!} \mathbf{B}^n t^n \right\} = e^t \{\mathbf{I} + \mathbf{B}t\} \\ &= e^t \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \right\} = \begin{pmatrix} e^t & 0 \\ 2te^t & e^t \end{pmatrix}, \end{aligned}$$

and the complete solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 2te^t & e^t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

Example 1.11 *Find the complete solution of the system*

$$y_1' = y_2 + y_3,$$

$$y_2' = y_1 + y_3,$$

$$y_3' = y_1 + y_2.$$

Here we also have a couple of solution possibilities.

1) The system can also be written

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

so the **eigenvalues** are the roots of the polynomial

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 1 + 1 + \lambda + \lambda + \lambda = -(\lambda^3 - 3\lambda - 2).$$

We immediately **guess** the roots $\lambda = -1$ and $\lambda = 2$. Then we get by a reduction,

$$-(\lambda^3 - 3\lambda - 2) = -(\lambda+1)(\lambda-2)(\lambda+1) = -(\lambda+1)^2(\lambda-2),$$

so $\lambda_1 = \lambda_2 = -1$ is a root of multiplicity w, and $\lambda_3 = 2$ is a simple root.

If $\lambda = -1$, we get the following system of equations for the eigenvectors,

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\mathbf{v} = \mathbf{0}.$$

Two linearly independent vectors which satisfy these equations are e.g.

$$\mathbf{v}_1 = (2, -1, -1) \qquad \text{and} \qquad \mathbf{v}_2 = (1, 1, -2).$$

If $\lambda = 2$ then we get

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\mathbf{v},$$

and we can e.g. choose the solution $\mathbf{v}_3 = (1, 1, 1)$. The complete solution is

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{-t} & e^{-t} & e^{2t} \\ -e^{-t} & e^{-t} & e^{2t} \\ -e^{-t} & -2e^{-t} & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

where c_1, c_2 and c_3 are arbitrary constants.

2) The **"fumbling method"**. It follows immediately of the symmetry of the equations that

$$\frac{d}{dt}(y_1 - y_2) = -(y_1 - y_2), \qquad \text{thus } y_1 - y_2 = 3c_1 e^{-t},$$

$$\frac{d}{dt}(y_2 - y_3) = -(y_2 - y_3), \qquad \text{thus } y_2 - y_3 = 3c_2 e^{-t},$$

hence by addition $y_1 - y_3 = 3(c_1 + c_2)e^{-t}$. Finally,

$$\frac{d}{dt}(y_1 + y_2 + y_3) = 2(y_1 + y_2 + y_3), \qquad \text{thus } y_1 + y_2 + y_3 = 3c_3 e^{2t}.$$

Hence we get

$$\begin{cases} 2y_1 + y_3 = 3c_1 e^{-t} + 3c_3 e^{2t}, \\ y_1 - y_3 = 3(c_1 + c_2)e^{-t}, \end{cases}$$

i.e.

$$\begin{aligned} y_1 &= (2c_1 + c_2)e^{-t} + c_3 e^{2t}, \\ y_2 &= (-c_1 + c_2)e^{-t} + c_3 e^{2t}, \\ y_3 &= (-c_1 - 2c_2)e^{-t} + c_3 e^{2t}, \end{aligned}$$

or written in a different way,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where c_1, c_2 and c_3 are arbitrary constants.

Example 1.12 *Find the complete solution of the system of differential equations*

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y},$$

where

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 4 \\ -1 & -1 & 0 \\ -2 & 0 & -3 \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}.$$

The **eigenvalues** are the roots of the polynomial

$$\begin{aligned} \begin{vmatrix} 3-\lambda & 0 & 4 \\ -1 & -1-\lambda & 0 \\ -2 & 0 & -3-\lambda \end{vmatrix} &= (-1-\lambda)\begin{vmatrix} 3-\lambda & 4 \\ -2 & -3-\lambda \end{vmatrix} \\ &= -(\lambda+1)\{\lambda^2 - 9 + 8\} = -(\lambda-1)(\lambda+1)^2. \end{aligned}$$

The eigenvalues are the simple root $\lambda = 1$ and $\lambda = -1$ of multiplicity 2.

The eigenvectors (a, b, c) are determined by the equation

$$\begin{cases} 3a + 4c = \lambda a, \\ -a - b = \lambda b, \\ -2a - 3c = \lambda c. \end{cases}$$

If $\lambda = 1$, then

$$\left\{\begin{array}{r} 2a+4c=0,\\ a+2b=0,\\ -2a-4c=0, \end{array}\right. \qquad \text{thus } \left\{\begin{array}{l} a=-2c=-2b,\\ (a,b,c)=c(-2,1,1). \end{array}\right.$$

If $\lambda = -1$, then

$$\left\{\begin{array}{r} 4a+4c=0,\\ -a=0,\\ -2a-2c=0, \end{array}\right. \qquad \text{thus } a=c=0, \text{ and } b \text{ is a free parameter.}$$

Thus we have found two linearly independent solutions. The third solution must have the structure

$$\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix} = \begin{pmatrix} a_1e^{-t}+a_2te^{-t}\\ b_1e^{-t}+b_2te^{-t}\\ c_1e^{-t}+c_2te^{-t}\end{pmatrix},$$

where

$$\frac{d}{dt}\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix} = \begin{pmatrix} (-a_1+a_2)e^{-t}-a_2te^{-t}\\ (-b_1+b_2)e^{-t}-b_2te^{-t}\\ (-c_1+c_2)e^{-t}e^{-t}-c_2te^{-t}\end{pmatrix},$$

and

$$\begin{pmatrix} 3 & 0 & 4\\ -1 & -1 & 0\\ -2 & 0 & -3\end{pmatrix}\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix} = \begin{pmatrix} (3a_1+4c_1)e^{-t}+(3a_2+4c_2)te^{-t}\\ (-a_1-b_1)e^{-t}+(-a_2-b_2)te^{-t}\\ (-2a_1-3c_1)e^{-t}+(-2a_2-3c_2)te^{-t}\end{pmatrix}.$$

We get by identifying the coefficients that

$$\begin{array}{ll} 3a_1+4c_1=-a_1+a_2, & \text{thus } 4c_1=-4a_1+a_2,\\ -a_1-b_1=-b_1+b_2, & \text{thus } b_2=-a_1,\\ -2a_1-3c_1=-c_1+c_2, & \text{thus } 2c_1+c_2=-2a_1,\\ 3a_2+4c_2=-a_2, & \text{thus } c_2=-a_2,\\ -a_2-b_2=-b_2, & \text{thus } a_2=0,\\ -2a_2-3c_2=-c_2, & \text{thus } c_2=-a_2. \end{array}$$

It follows from $a_2=0$ that $c_2=0$, hence $c_1=-a_1=b_2$. Finally, b_1 can be chosen freely.

The complete solution is

$$\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix} = c_1e^t\begin{pmatrix} -2\\ 1\\ 1\end{pmatrix} + c_2e^{-t}\begin{pmatrix} 0\\ 1\\ 0\end{pmatrix} + c_3e^{-t}\begin{pmatrix} 1\\ t\\ -1\end{pmatrix} = \begin{pmatrix} -2=e^t & 0 & e^{-t}\\ e^t & e^{-t} & te^{-t}\\ e^t & 0 & -e^{-t}\end{pmatrix}\begin{pmatrix} c_1\\ c_2\\ c_3\end{pmatrix},$$

where c_1, c_2 and c_3 are arbitrary constants.

Example 1.13 *Find the complete solution of the system*

$$\mathbf{Y}' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \mathbf{Y}.$$

The matrix is an upper triangular matrix, so it follows immediately by inspection that the two eigenvalues $\lambda = \pm 1$ both have multiplicity 2. It also follows immediately that y_4 and y_3 must have the simplified structure

$$y_4 = ke^{-t} \qquad \text{and} \qquad y_3 = c_3 e^{-t} + c_4 t e^{-t}.$$

We conclude that the general structure of solution must be

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} a_1 e^t + a_2 t e^t + a_3 e^{-t} + a_4 t e^{-t} \\ b_1 e^t + b_2 t e^t + b_3 e^{-t} + b_4 t e^{-t} \\ c_3 e^{-t} + c_4 t e^{-t} \\ k e^{-t} \end{pmatrix}.$$

Since

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} (a_1+a_2)e^t+a_2te^t+(-a_3+a_4)e^{-t}-a_4te^{-t} \\ (b_1+b_2)e^t+b_2te^t+(-b_3+b_4)e^{-t}-b_4te^{-t} \\ (-c_3+c_4)e^{-t}-c_4te^{-t} \\ -ke^{-t} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} (a_1+b_1)e^t+(a_2+b_2)te^t+(a_3+b_3+c_3+k)e^{-t}e^{-t}+(a_4+b_4+c_4)te^{-t} \\ b_1e^t+b_2te^t+(b_3+2c_3+k)e^{-t}+(b_4+2c_4)te^{-t} \\ (-c_3+k)e^{-t}-c_4te^{-t} \\ -ke^{-t} \end{pmatrix},$$

we conclude by identifying the coefficients that

$$\begin{cases} a_1+b_1=a_1+a_2, \\ b_1=b_1+b_2, \end{cases} \qquad \begin{cases} a_2+b_2=a_2, \\ b_2=b_2, \end{cases}$$

and

$$\begin{cases} a_3+b_3+c_3+k=-a_3+a_4, \\ b_3+2c_3+k=-b_3+b_4, \\ -c_3+k=-c_3+c_4, \end{cases} \qquad \begin{cases} a_4+b_4+c_4=-a_4, \\ b_4+2c_4=-b_4, \\ -c_4=-c_4. \end{cases}$$

It follows immediately from these equations that

$$b_2=0, \qquad b_4=-c_4=-k, \qquad b_1=a_2.$$

Then the equations are reduced to

$$\begin{cases} b_3+c_3=-2a_3+a_4-k, \\ 2b_3+2c_3=-2k, \\ k=-2a_4+k, \end{cases}$$

hence

$$b_3+c_3=-k, \qquad a_4=0=a_3, \qquad \text{thus } c_3=-k-b_3.$$

Let the free parameters be a_1, a_2, b_3 and k. Then

$$a_3=a_4=0, \quad b_1=a_2, \quad b_2=0, \quad b_4=-k, \quad c_3=-k-b_3, \quad c_4=k.$$

The complete solution is

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} &= \begin{pmatrix} a_1e^t+a_2te^t \\ a_2e^t+b_3e^{-t}-kte^{-t} \\ (-k-b_3)e^{-t}+kte^{-t} \\ ke^{-t} \end{pmatrix} \\ &= a_1\begin{pmatrix} e^t \\ 0 \\ 0 \\ 0 \end{pmatrix}+a_2\begin{pmatrix} te^t \\ e^t \\ 0 \\ 0 \end{pmatrix}+b_3\begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \\ 0 \end{pmatrix}+k\begin{pmatrix} 0 \\ -te^{-t} \\ (t-1)e^{-t} \\ e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} e^t & te^t & 0 & 0 \\ 0 & e^t & e^{-t} & -te^{-t} \\ 0 & 0 & -e^{-t} & (t-1)e^{-t} \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ b_3 \\ k \end{pmatrix}, \end{aligned}$$

where a_1, a_2, b_3 and k are arbitrary constants.

Example 1.14 *Find the complete solution of the homogeneous system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

The characteristic polynomial

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (\lambda-1)^2-4 = (\lambda-1)^2-2^2 = (\lambda+1)(\lambda-3)$$

has the roots $\lambda_1 = -1$ and $\lambda_3 = 3$.

An eigenvector corresponding to an eigenvalue λ is a cross vector of

$$(1-\lambda, 1)$$

[first row in the matrix $\mathbf{A} - \lambda\mathbf{I}$].
If $\lambda_1 = -1$, then e.g. $\mathbf{v}_1 = (1,-2)^T$.
If $\lambda_3 = 3$, then e.g. $\mathbf{v}_2 = (1,2)^T$.
The complete solution is

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} = \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_1e^{-t} + c_2e^{3t} \\ -2c_1e^{-t} + 3c_2e^{3t} \end{pmatrix}, \qquad t \in \mathbb{R}, \end{aligned}$$

where c_1 and c_2 are arbitrary constants, and where we have indicated three equivalent results.

Example 1.15 *Given* $\mathbf{x} = (x_1, x_2)^T$, $\dfrac{d\mathbf{x}}{dt} = \left(\dfrac{dx_1}{dt}, \dfrac{dx_2}{dt}\right)^T$, *and*

$$\mathbf{A} = \begin{pmatrix} -7 & 2 \\ -36 & 10 \end{pmatrix}.$$

Find that solution of the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x}, \qquad t \in \mathbb{R},$$

for which $\mathbf{x}(0) = (1,5)^T$.

The characteristic polynomial

$$\begin{vmatrix} -\lambda-7 & 2 \\ -36 & 10-\lambda \end{vmatrix} = (\lambda+7)(\lambda-10)+72 = \lambda^2-3\lambda+2$$

has the roots $\lambda_1 = 1$ and $\lambda_2 = 2$.
If $\lambda_1 = 1$, then an eigenvector is a cross vector of $(-8,2)$, e.g. $\mathbf{v}_1 = (1,4)^T$.

If $\lambda_2 = 2$, then an eigenvector is a cross vector of $(-9, 2)$, e.g. $\mathbf{v}_2 = (2, 9)^T$.
The complete solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ 9 \end{pmatrix} e^{2t}.$$

We get for $t = 0$,

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ 4c_1 + 9c_2 \end{pmatrix},$$

hence $c_1 = -1$ and $c_2 = 1$.
The particular solution is then given by

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -e^t + 2e^{2t} \\ -4e^t + 9e^{2t} \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Example 1.16 *Find the complete solution of the homogeneous system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

The characteristic polynomial

$$\begin{vmatrix} -3-\lambda & 1 \\ -1 & -3-\lambda \end{vmatrix} = (\lambda+3)^2 + 1$$

has the complex conjugated roots $a \pm i\omega = -3 \pm 1 \cdot i$.
A complex eigenvector $\alpha + i\beta$ corresponding to $-3 + i$ is a cross vector to $(-i, 1)$, e.g.

$$\alpha + i\beta = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then a fundamental matrix is given by

$$\begin{aligned} \mathbf{\Phi}(t) &= e^{at}\cos\omega t(\alpha \;\; \beta) + e^{at}\sin\omega t(-\beta \;\; \alpha) = e^{-3t}\cos t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^{-3t}\sin t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= e^{-3t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \end{aligned}$$

The complete solution is then

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-3t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^{-3t}(c_1\cos t + c_2 \sin t) \\ e^{-3t}(-c_1 \sin t + c_2 \cos t) \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

Example 1.17 *Find the complete solution of the homogeneous system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = (\lambda-1)(\lambda-5)-12 = \lambda^2-6\lambda-7 = (\lambda-3)^2-16 = (\lambda-7)(\lambda+1),$$

so the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 7$.

Once the characteristic polynomial has been found, there are several ways to continue. We shall here give some variants.

First variant. The eigenvalue method. The eigenvector corresponding to an eigenvalue λ is a cross vector to $(1-\lambda, 3)$.
If $\lambda_1 = -1$, then we e.g. get $\mathbf{v}_1 = (3,-2)^T$.
If $\lambda_2 = 7$, then we e.g. get $\mathbf{v}_2 = (1,2)^T$.
The complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 3 \\ -2 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{7t} = \begin{pmatrix} 3e^{-t} & e^{7t} \\ -2e^{-t} & 2e^{7t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad t \in \mathbb{R},$$

where c_1 and c_2 are arbitrary constants.

Second variant. Discussion of the structure of the solution. The solution must necessarily have the structure

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ae^{-t} + be^{7t} \\ ce^{-t} + de^{7t} \end{pmatrix},$$

where we shall eliminate two of the parameters. We first calculate

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -ae^{-t} + 7be^{7t} \\ -ce^{-t} + 7de^{7t} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} ae^{-t} + be^{7t} \\ ce^{-t} + de^{7t} \end{pmatrix} = \begin{pmatrix} (a+3c)e^{-t} + (b+3d)e^{7t} \\ (4a+5c)e^{-t} + (4b+5b)e^{7t} \end{pmatrix}.$$

Now, e^{-t} and e^{7t} are linearly independent, so we get by an identification of the coefficients that

$$\begin{cases} -a = a+3c, \\ -c = 4a+3c, \end{cases} \qquad \text{og} \qquad \begin{cases} 7b = b+3d, \\ 7d = 4b+5b, \end{cases}$$

hence $2a + 3c = 0$ and $2b = d$.

It follows that we may choose $a = 3$, $c = -2$, and $b = 1$, $d = 2$, and then we obtain the complete solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} 3 \\ -2 \end{pmatrix} + c_2 e^{7t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3e^{-t} & e^{7t} \\ -2e^{-t} & 2e^{7t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad t \in \mathbb{R},$$

where c_1 and c_2 are arbitrary constants.

Third variant. The fumbling method. We expand the system,

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + 3x_2, \qquad \text{dvs. } x_2 = \dfrac{1}{3}\dfrac{dx_1}{dt} - \dfrac{1}{3}x_1, \\ \dfrac{dx_2}{dt} = 4x_1 + 5x_2. \end{cases}$$

Here we eliminate x_2,

$$\frac{dx_2}{dt} = \frac{1}{3}\frac{d^2x_1}{dt^2} - \frac{1}{3}\frac{dx_1}{dt} = 4x_1 + 5x_2 = \frac{5}{3}\frac{dx_1}{dt} + \left(4 - \frac{5}{3}\right)x_1,$$

hence by a reduction,

$$\frac{d^2x_1}{dt^2} - 6\frac{dx_1}{dt} - 7x_1 = 0.$$

The characteristic equation $R^2 - 6R - 7 = 0$ has the roots $R = -1$ and $R = 7$, so

$$x_1 = ae^{-t} + be^{7t},$$

hence by putting this into the first equation,

$$x_2 = \frac{1}{3}\left(\frac{dx_1}{dt} - x_1\right) = \frac{1}{3}\left(-ae^{-t} + 7be^{7t} - ae^{-t} - be^{7t}\right) = -\frac{2}{3}ae^{-t} + 2be^{7t}.$$

If we write $c_1 = \dfrac{a}{3}$ and $c_2 = b$, the complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ae^{-t} + be^{7t} \\ -\frac{2}{3}ae^{-t} + 2be^{7t} \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 3 \\ -2 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Fourth variant. The exponential matrix. This is given by a formula,

$$\begin{aligned} \exp(\mathbf{A}t) &= \frac{-\lambda_2 e^{\lambda_1 t} + \lambda_1 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\mathbf{I} + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\mathbf{A} = -\frac{1}{8}\{-7e^{-t} - e^{7t}\}\mathbf{I} - \frac{1}{8}\{e^{-t} - e^{7t}\}\mathbf{A} \\ &= \frac{1}{8}\begin{pmatrix} 7e^{-t}+e^{7t} & 0 \\ 0 & 7e^{-t}+e^{7t} \end{pmatrix} + \frac{1}{8}\begin{pmatrix} e^{7t}-e^{-t} & 3e^{7t}-3e^{-t} \\ 4e^{7t}-4e^{-t} & 5e^{7t}-5e^{-t} \end{pmatrix} \\ &= \frac{1}{8}\begin{pmatrix} 6e^{-t} + 2e^{7t} & -3e^{-t} + 3e^{7t} \\ -4e^{-t} + 4e^{7t} & 2e^{-t} + 6e^{7t} \end{pmatrix}. \end{aligned}$$

Here $\dfrac{1}{8}$ can be built into the arbitrary constants, so the complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6e^{-t} + 2e^{7t} & -3e^{-t} + 3e^{7t} \\ -4e^{-t} + 4e^{7t} & 2e^{-t} + 6e^{7t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Fifth variant. (Sketch). It is also to find the exponential matrix by using its structure

$$\exp(\mathbf{A}t) = \varphi(t)\mathbf{I} + \psi(t)\mathbf{A}, \qquad \varphi(0) = 1, \quad \psi(0) = 0,$$

and by checking the matrix differential equation,

$$\frac{d}{dt}\exp(\mathbf{A}t) = \mathbf{A}\,\exp(\mathbf{A}t)$$

and finally apply Caley-Hamilton's equation,

$$\mathbf{A}^2 - 6\mathbf{A} - 7\mathbf{I} = \mathbf{0}, \qquad \text{dvs. } \mathbf{A}^2 = 6\mathbf{A} + 7\mathbf{I}.$$

However, if one does not use some clever calculational tricks, one may easily end up in a mess of formulæ, so this variant is not given here in all its details.

Example 1.18 *Find the complete solution of the homogeneous system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Here the eigenvalue method is the simplest method.

The **eigenvalues** are the roots of the equation

$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 3^2 = (\lambda-5)(\lambda+1) = 0,$$

thus $\lambda = 5$ or $\lambda = -1$.

The **eigenvectors**. An eigenvector $\mathbf{v}$ is a cross vector to $2-\lambda, 3)$
If $\lambda = 5$, then we get a cross vector $(-3, 3)$, so we may choose $\mathbf{v}_1 = (1, 1)$.
If $\lambda = -1$, then we get the cross vector $(3, 3)$, and we may choose $\mathbf{v}_2 = (1, -1)$.
The **complete solution** is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{5t} & e^{-t} \\ e^{5t} & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where c_1, c_2 are arbitrary constants.

Example 1.19 *Find the complete solution of the homogeneous system*

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

First we find the eigenvalues of the matrix:

$$\begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (\lambda-1)(\lambda-2)-12 = \lambda^2-3\lambda-10 = 0,$$

hence the eigenvalues are $\lambda = -2$ and $\lambda = 5$.
If $\lambda = -2$, then $\mathbf{v}_1 = (1, 1)$ is an eigenvector.
If $\lambda = 5$, then $\mathbf{v}_2 = (3, 4)$ is an eigenvector.
The complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} e^{-2t} & 3e^{5t} \\ -e^{-2t} & 4e^{5t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Example 1.20 *Find the complete solution of the homogeneous system*

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We shall here demonstrate three variants.

1) The **eigenvalue method**. The eigenvalues are the roots of the characteristic polynomial

$$\begin{vmatrix} 1-\lambda & 5 \\ 1 & -3-\lambda \end{vmatrix} = (\lambda-1)(\lambda+3)-5 = \lambda^2+2\lambda-8 = (\lambda+1)^2-9,$$

hence

$$\lambda = -1 \pm 3 = \begin{cases} 2, \\ -4. \end{cases}$$

a) If $\lambda = 2$, then we get the matrix

$$\begin{pmatrix} 1-\lambda & 5 \\ 1 & -3-\lambda \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 1 & -5 \end{pmatrix},$$

and we conclude that we may choose $(5, 1)$ as an eigenvector.

b) If $\lambda = -4$, then we get the matrix

$$\begin{pmatrix} 1-\lambda & 5 \\ 1 & -3-\lambda \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 1 & 1 \end{pmatrix},$$

and we choose e.g. $(1,-1)$ as an eigenvector.

Summing up, the complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5e^{2t} & e^{-4t} \\ e^{2t} & -e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

2) The **fumbling method**. We expand the system of equations,

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + 5x_2, \\ \dfrac{dx_2}{dt} = x_1 - 3x_2. \end{cases}$$

It follows from the latter equation that

(5) $x_1 = \dfrac{dx_2}{dt} + 3x_2,$

hence by insertion into the former one,

$$\frac{dx_1}{dt} = \frac{d^2x_2}{dt^2} + 3\frac{dx_2}{dt} = x_1 + 5x_2 = \frac{dx_2}{dt} + 8x_2.$$

Then by a rearrangement

$$\frac{d^2x_2}{dt^2} + 2\frac{dx_2}{dt} - 8x_2 = 0.$$

The characteristic equation $R^2 + 2R - 8 = 0$ has the roots $R = 2$ and $R = -4$, so

$$x_2 = c_2e^{2t} + c_2e^{-4t}.$$

If we put this into (5), we get

$$x_1 = \frac{dx_2}{dt} + 3x_2 = (2c_1e^{2t} - 4c_2e^{-4t}) + (3c_1e^{2t} + 3c_2e^{-4t}) = 5c_1e^{2t} - c_2e^{-4t}.$$

Summing up we get

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5c_1e^{2t} - c_2e^{-4t} \\ c_1e^{2t} + c_2e^{-4t} \end{pmatrix} = \begin{pmatrix} 5e^{2t} & -e^{-4t} \\ e^{2t} & e^{-4t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

3) The **exponential matrix**. The characteristic polynomial is

$$(\lambda+1)^2 - 9.$$

Then by **Caley-Hamilton's theorem**,

$$(\mathbf{A}+\mathbf{I})^2 - 9\mathbf{I} = \mathbf{0}, \quad \text{dvs. } \mathbf{B}^2 = 9\mathbf{I}, \text{ hvor } \mathbf{B} = \mathbf{A}+\mathbf{I}.$$

Since $\mathbf{I}$ and $\mathbf{A}$ commute, we have

$$\begin{aligned}
\exp(\mathbf{A}t) &= \exp((\mathbf{B}-\mathbf{I})t) = e^{-t}\exp(\mathbf{B}t) \\
&= e^{-t}\left\{\sum_{n=0}^{\infty}\frac{1}{(2n)!}\,\mathbf{B}^{2n}t^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,\mathbf{B}^{2n+1}t^{2n+1}\right\} \\
&= e^{-t}\left\{\sum_{n=0}^{\infty}\frac{(3t)^{2n}}{(2n)!}\,\mathbf{I} + \frac{1}{3}\sum_{n=0}^{\infty}\frac{(3t)^{2n+1}}{(2n+1)!}\,\mathbf{B}\right\} \\
&= e^{-t}\left\{\cosh(3t)\mathbf{I} + \frac{1}{3}\sinh(3t)\mathbf{B}\right\} \\
&= e^{-t}\left\{\cosh(3t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3}\sinh(3t)\begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}\right\} \\
&= \frac{1}{3}\,e^{-t}\begin{pmatrix} 3\cosh 3t + 2\sinh 3t & 5\sinh 3t \\ \sinh 3t & 3\cosh 3t - 2\sinh 3t \end{pmatrix} \\
&= \frac{1}{6}\,e^{-t}\begin{pmatrix} 3e^{3t}+3e^{-3t}+2e^{3t}-2e^{-3t} & 5e^{3t}-5e^{-3t} \\ e^{3t}-e^{-3t} & 3e^{3t}+3e^{-3t}-2e^{3t}+2e^{-3t} \end{pmatrix} \\
&= \frac{1}{6}\,e^{-t}\begin{pmatrix} 5e^{3t}+e^{-3t} & 5e^{3t}-5e^{-3t} \\ e^{3t}-e^{-3t} & e^{3t}+5e^{-3t} \end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix} 5e^{2t}+e^{-4t} & 5e^{2t}-5e^{-4t} \\ e^{2t}-e^{-4t} & e^{2t}+5e^{-4t} \end{pmatrix}.
\end{aligned}$$

Thus the complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 5e^{2t}+e^{-4t} \\ e^{2t}-e^{-4t} \end{pmatrix} + c_2 \begin{pmatrix} 5e^{2t}-5e^{-4t} \\ e^{2t}+5e^{-4t} \end{pmatrix}.$$

Example 1.21 *Find the complete solution of the homogeneous system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

.

We shall here only apply the eigenvalue method, even if other methods may also be applied.

The characteristic polynomial

$$\begin{vmatrix} 4-\lambda & 3 \\ 3 & -4-\lambda \end{vmatrix} = \lambda^2 - 25$$

has the roots $\lambda = \pm 5$.

If $\lambda = 5$, then

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 4-5 & 3 \\ 3 & -4-5 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix},$$

hence $(3,1)$ is an eigenvector corresponding to $\lambda = 5$.
If $\lambda = -5$, then

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 4+5 & 3 \\ 3 & -4+5 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix},$$

hence $(1,-3)$ is an eigenvector corresponding to $\lambda = -5$.
The complete solution is

$$\mathbf{x}(t) = c_1 e^{5t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3e^{5t} & e^{-5t} \\ e^{5t} & -3e^{-5t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

for $t \in \mathbb{R}$, where c_1 and c_2 are arbitrary constants.

Example 1.22 *Find the complete solution of the homogeneous system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -3 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

It follows from

$$\begin{vmatrix} 1-\lambda & 2 \\ -3 & 8-\lambda \end{vmatrix} = (1-\lambda)(8-\lambda)+6 = \lambda^2-9\lambda+14 = (\lambda-7)(\lambda-2),$$

that the eigenvalues are $\lambda = 2$ and $\lambda = 7$.

1) If $\lambda = 2$, then an eigenvector is a cross vector to $(1-\lambda, 2) = (-1, 2)$, so we get e.g. $(2, 1)$ as an eigenvector.

2) If $\lambda = 7$, then an eigenvector is a cross vector to $(1-\lambda, 2) = (-6, 2)$, so we get e.g. $(1, 3)$ as an eigenvector.

The complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Example 1.23 *Find the complete solution of the homogeneous system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} 4-\lambda & 6 \\ 8 & 2-\lambda \end{vmatrix} = (\lambda-4)(\lambda-2)-48 = \lambda^2-6\lambda-40 = (\lambda-3)^2 - 7^2 = 0.$$

We get the two eigenvalues

$$\lambda = 3 \pm 7 = \begin{cases} 10, \\ -4. \end{cases}$$

An eigenvector corresponding to $\lambda = 10$ is a cross vector to $(4-10, 6)^T = (-6, 6)^T$, e.g. $(1, 1)$.
An eigenvector corresponding to $\lambda = -4$ is a cross vector to $(4-(-4), 6)^T = (8, 6)^T$, e.g. $(3, -4)$.
The complete solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{10t} + c_2 \begin{pmatrix} 3 \\ -4 \end{pmatrix} e^{-4t} = \begin{pmatrix} c_1 e^{10t} + 3c_2 e^{-4t} \\ c_1 e^{10t} - 4c_2 e^{-4t} \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

Example 1.24 *Given the matrix* $\mathbf{A}$ *by*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}.$$

Find $\exp(\mathbf{A}t) = e^{\mathbf{A}t}$.

The characteristic polynomial

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{vmatrix} = (\lambda-1)(\lambda+4)-6 = \lambda^2+3\lambda-10 = (\lambda-2)(\lambda+5)$$

has the simple roots $\lambda = 2$ and $\lambda = -5$. Then we have two methods:

1) **Definition of the exponential matrix**. Since $\mathbf{I}$ and $\mathbf{A}$ commute, we get

$$\exp(\mathbf{A}t) = \exp((\mathbf{A}-2\mathbf{I})t+2t\mathbf{I}) = e^{2t}\exp(\mathbf{B}t),$$

where

$$\mathbf{B} = \mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}$$

and

$$\mathbf{B}^2 = \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} 7 & -14 \\ -21 & 42 \end{pmatrix} = -7\mathbf{B}.$$

It follows by induction that $\mathbf{B}^n = (-7)^{n-1}\mathbf{B}$, $n \in \mathbb{N}$. Then

$$\begin{aligned}
\exp(\mathbf{A}t) &= e^{2t}\exp(\mathbf{B}t) = e^{2t}\left\{\mathbf{I} + \sum_{n=1}^{\infty}\frac{1}{n!}\mathbf{B}^n t^n\right\} = e^{2t}\left\{\mathbf{I} + \sum_{n=1}^{\infty}\frac{(-7)^{n-1}}{n!}t^n\cdot\mathbf{B}\right\} \\
&= e^{2t}\left\{\mathbf{I} - \frac{1}{7}\sum_{n=1}^{\infty}\frac{1}{n!}(-7t)^n\mathbf{B}\right\} = e^{2t}\left\{\mathbf{I} - \frac{1}{7}(e^{-7t}-1)\mathbf{B}\right\} \\
&= \frac{1}{7}\begin{pmatrix} 6e^{2t}+e^{-5t} & 2e^{2t}-2e^{-5t} \\ 3e^{2t}-3e^{-5t} & e^{2t}+6e^{-5t} \end{pmatrix}.
\end{aligned}$$

2) The **eigenvalue method**. We have previously found the eigenvalues $\lambda = 2$ and $\lambda = -5$. We choose an eigenvector as a cross vector to

$$(1-\lambda, 2) \qquad \text{or to} \qquad (3, -4-\lambda).$$

To $\lambda = 2$ corresponds e.g. the eigenvector $\mathbf{v}_1 = (2, \lambda - 1) = (2, 1)$.

To $\lambda = -5$ corresponds e.g. the eigenvector $\mathbf{v}_2 = (-4-\lambda, -3) = (1, -3)$.

Then a fundamental matrix is

$$\mathbf{\Phi}(t) = \left(e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad e^{-5t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} \right) = \begin{pmatrix} 2e^{2t} & e^{-5t} \\ e^{2t} & -3e^{-5t} \end{pmatrix}.$$

Now,

$$\mathbf{\Phi}(0) = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \quad \text{og} \quad \mathbf{\Phi}(0)^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix},$$

so the exponential matrix is

$$\exp(\mathbf{A}t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1} = \begin{pmatrix} 2e^{2t} & e^{-5t} \\ e^{2t} & -3e^{-5t} \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\frac{1}{7} = \frac{1}{7}\begin{pmatrix} 6e^{2t}+e^{-5t} & 2e^{2t}-2e^{-5t} \\ 3e^{2t}-3e^{-5t} & e^{2t}+6e^{-5t} \end{pmatrix}.$$

Example 1.25 *Find the complete solution of the homogeneous system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

We shall here give four variants.

1) The **fumbling method**. In the actual case this is the simplest variant. It follows immediately from the system of equations that

$$\frac{dx_1}{dt} = \frac{dx_2}{dt} = \frac{dx_3}{dt} = x_1 + x_2 + x_3,$$

hence (by some conveniently chosen constants)

$$x_2 = x_1 + 3c_2, \qquad x_3 = x_1 + 3c_3,$$

and

$$\frac{d}{dt}(x_1 + x_2 + x_3) = 3(x_1 + x_2 + x_3).$$

We obtain from these equations that

$$x_1 + x_2 + x_3 = 3x_1 + 3c_2 + 3c_3 = 3c_1 e^{3t},$$

hence

$$\begin{cases} x_1 = c_1e^{3t}-c_2-c_3, \\ x_2 = c_1e^{3t}+2c_2-c_3, \\ x_3 = c_1e^{3t}-c_2+2c_3, \end{cases}$$

and thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1e^{-3t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

2) The **standard method**. The eigenvalues of the matrix are the solutions of the equation

$$0 = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3+2-3(1-\lambda) = -\lambda^3+3\lambda^2 = -\lambda^2(\lambda-3).$$

It follows that $\lambda = 3$ is a simple root and that $\lambda = 0$ is a double root. Since the matrix $\mathbf{A}$ is symmetric, its algebraic multiplicity is equal to its geometric multiplicity for $\lambda = 0$.

Let $\mathbf{y} = (y_1, y_2, y_3)$ be an eigenvector corresponding to the eigenvalue $\lambda = 3$, thus

$$3\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1+y_2+y_3 \\ y_1+y_2+y_2 \\ y_1+y_2+y_3 \end{pmatrix}.$$

It follows immediately that $y_1 = y_2 = y_3$, so we may choose $(1,1,1)$ as an eigenvector.

If $\lambda = 0$ we get analogously the condition

$$y_1+y_2+y_3 = 0,$$

which describes a plane in space. We shall only choose two linearly independent vectors, the coordinates of which satisfy this condition. This may be done in several ways. If we e.g. choose $(1,-1,0)$ and $(1,0,-1)$, then we get the complete solution

$$\mathbf{x}(t) = c_1e^{3t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

3) **Calculation of the exponential matrix**. It follows immediately that

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{A}^2 = 3\mathbf{A}, \dots, \mathbf{A}^n = 3^{n-1}\mathbf{A},$$

so by insertion into the exponential series,

$$\begin{aligned} \exp(\mathbf{A}t) &= \sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{A}^nt^n = \mathbf{I}+\left\{\sum_{n=1}^{\infty}\frac{1}{n!}3^{n-1}t^n\right\}\mathbf{A} \\ &= \mathbf{I}+\frac{1}{3}\left\{1+\sum_{n=1}^{\infty}\frac{1}{n!}(3t)^n-1\right\}\mathbf{A} = \mathbf{I}+\frac{1}{3}(e^{3t}-1)\mathbf{A} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} e^{3t}-1 & e^{3t}-1 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}-1 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}-1 & e^{3t}-1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} e^{3t}+2 & e^{3t}-1 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}+2 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}-1 & e^{3t}+2 \end{pmatrix}. \end{aligned}$$

The complete solution is all linear combinations of the columns of the exponential matrix,

$$\mathbf{x}(t) = c_1 \begin{pmatrix} e^{3t}+2 \\ e^{3t}-1 \\ e^{3t}-1 \end{pmatrix} + c_2 \begin{pmatrix} e^{3t}-1 \\ e^{3t}+2 \\ e^{3t}-1 \end{pmatrix} + c_3 \begin{pmatrix} e^{3t}-1 \\ e^{3t}-1 \\ e^{3t}+2 \end{pmatrix}.$$

Remark 1.1 The three solutions are of course equivalent, even though the constants do not correspond here.

4) **Cayley-Hamilton's theorem**. We prove as in (2) that the characteristic polynomial is

$$(-1)^3 \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3).$$

The corresponding differential equation

$$\frac{d^3x}{dt^3} - 3\frac{d^2x}{dt^2} = 0$$

has the complete solution $x(t)$, where

$$\begin{aligned} x(t) &= c_1 + c_2 t + c_3 e^{3t}, \\ x'(t) &= c_2 + 3c_3 e^{3t}, \\ x''(t) &= 9c_3 e^{3t}. \end{aligned}$$

The initial conditions are

$$x_i^{(j)}(0) = \delta_{ij} \qquad \text{for } i,\, j = 0,\, 1,\, 2.$$

If $i = 0$, then

$$\begin{cases} c_1 + c_3 = 1, \\ c_2 + 3c_3 = 0, \\ 9c_3 = 0, \end{cases} \quad \text{thus} \quad \begin{cases} c_1 = 1, \\ c_2 = 0, \\ c_3 = 0, \end{cases}$$

hence $x_0(t) = 1$.
If $i = 1$, then

$$\begin{cases} c_1 + c_3 = 0, \\ c_2 + 3c_3 = 1, \\ 9c_3 = 0, \end{cases} \quad \text{thus} \quad \begin{cases} c_1 = 0, \\ c_2 = 1, \\ c_3 = 0, \end{cases}$$

hence $x_1(t) = t$.
If $i = 2$, then

$$\begin{cases} c_1 + c_3 = 0, \\ c_2 + 3c_3 = 0, \\ 9c_3 = 1, \end{cases} \quad \text{thus} \quad \begin{cases} c_1 = -1/9, \\ c_2 = -1/3, \\ c_3 = 1/9, \end{cases}$$

hence

$$x_2(t) = -\frac{1}{9} - \frac{1}{3}\,t + \frac{1}{9}e^{3t}.$$

Then by **Caley-Hamilton's theorem** $\mathbf{A}^2 = 3\mathbf{A}$, and we get from the above that the exponential matrix is

$$\begin{aligned}\exp(\mathbf{A}t) &= x_0(t)\mathbf{I} + x_1(t)\mathbf{A} + x_2(t)\mathbf{A}^2 = \mathbf{I} + t\mathbf{A} + \left(-\frac{1}{9} - \frac{1}{3}\,t + \frac{1}{9}\,e^{3t}\right) 3\mathbf{A} \\ &= \mathbf{I} + \frac{1}{3}(e^{3t}-1)\mathbf{A} = \frac{1}{3}\{3\mathbf{I} + (e^{3t}-1)\mathbf{A}\} = \frac{1}{3}\begin{pmatrix} e^{3t}+2 & e^{3t}-1 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}+2 & e^{3t}-1 \\ e^{3t}-1 & e^{3t}-1 & e^{3t}+2 \end{pmatrix}.\end{aligned}$$

The complete solution of the differential equation is composed of all linear combinations of the columns, i.e.

$$\mathbf{x}(t) = c_1 \begin{pmatrix} e^{3t}+2 \\ e^{3t}-1 \\ e^{3t}-1 \end{pmatrix} + c_2 \begin{pmatrix} e^{3t}-1 \\ e^{3t}+2 \\ e^{3t}-1 \end{pmatrix} + c_3 \begin{pmatrix} e^{3t}-1 \\ e^{3t}-1 \\ e^{3t}+2 \end{pmatrix},$$

where c_1, c_2, c_3 are arbitrary constants.

2 Inhomogeneous systems of linear differential equations

Example 2.1 *Find the complete solution of the system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

The **eigenvalues** are the roots of the polynomial

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda(\lambda+1)+1 = \lambda^2+\lambda+1,$$

so we have the complex conjugated eigenvalues

$$\lambda = -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}.$$

1) **Complex eigenvectors**. If $\lambda = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}$, then we get the matrix equation

$$\begin{pmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - i\,\frac{\sqrt{3}}{2} & -1 \\ 1 & -\frac{1}{2} - i\,\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A solution is a cross vector of e.g. the first row,

$$\left(+1, \frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\{(2,1) - i(0,\sqrt{3})\},$$

hence we can choose (multiply by 2),

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - i\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \qquad \text{for } \lambda_1 = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}.$$

Analogously we get

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \qquad \text{for } \lambda_2 = -\frac{1}{2} - i\,\frac{\sqrt{3}}{2}.$$

The complete *complex* solution of the *homogeneous equation* is

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \tilde{c}_1 e^{\lambda_1 t}\mathbf{v}_1 + \tilde{c}_2 e^{\lambda_2 t}\mathbf{v}_2 \\ &= \tilde{c}_1 e^{-t/2}\left\{\cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t\right\}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix} - i\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}\right\} \\ &\quad + \tilde{c}_2 e^{-t/2}\left\{\cos\frac{\sqrt{3}}{2}t - i\sin\frac{\sqrt{3}}{2}t\right\}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}\right\}. \end{aligned}$$

We get by splitting into the real and the imaginary part,

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \tilde{c}_1 e^{-t/2}\left\{\cos\frac{\sqrt{3}}{2}t\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \sin\frac{\sqrt{3}}{2}t\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}\right. \\ &\qquad \left. +i\left[\sin\frac{\sqrt{3}}{2}t\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \cos\frac{\sqrt{3}}{2}t\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}\right]\right\} \\ &\qquad +\tilde{c}_2 e^{-t/2}\left\{\cos\frac{\sqrt{3}}{2}t\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \sin\frac{\sqrt{3}}{2}t\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}\right. \\ &\qquad \left. -i\left[\sin\frac{\sqrt{3}}{2}t\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \cos\frac{\sqrt{3}}{2}t\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}\right]\right\}.\end{aligned}$$

We obtain the *real* complete solution by choosing $\overline{\tilde{c}}_2 = \tilde{c}_1$, hence with new (real) arbitrary constants we get the complete *real* solution of the homogeneous equation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-t/2}\begin{pmatrix} 2\cos\frac{\sqrt{3}}{2}t \\ \cos\frac{\sqrt{3}}{2}t + \sqrt{3}\cdot\sin\frac{\sqrt{3}}{2}t \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} 2\sin\frac{\sqrt{3}}{2}t \\ \sin\frac{\sqrt{3}}{2}t - \sqrt{3}\cdot\cos\frac{\sqrt{3}}{2}t \end{pmatrix}.$$

2) **Alternatively** one may only use real calculations. In fact, since $\lambda = -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}$, the complete solution of the homogeneous equation must have the structure

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t/2}\begin{pmatrix} a_1\cos\frac{\sqrt{3}}{2}t + a_2\sin\frac{\sqrt{3}}{2}t \\ b_1\cos\frac{\sqrt{3}}{2}t + b_2\sin\frac{\sqrt{3}}{2}t \end{pmatrix}.$$

We know that we have *two* arbitrary constants in the final solution, and here we have got *four* unknowns a_1, a_2, b_1, b_2, so we still have to eliminate two of them by means of the differential equation. Now,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t/2}\begin{pmatrix} \left(-\frac{1}{2}a_1+\frac{\sqrt{3}}{2}a_2\right)\cos\frac{\sqrt{3}}{2}t+\left(-\frac{\sqrt{3}}{2}a_1-\frac{1}{2}a_2\right)\sin\frac{\sqrt{3}}{2}t \\ \left(-\frac{1}{2}b_1+\frac{\sqrt{3}}{2}b_2\right)\cos\frac{\sqrt{3}}{2}+\left(-\frac{\sqrt{3}}{2}b_1-\frac{1}{2}b_2\right)\sin\frac{\sqrt{3}}{2}t \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t/2}\begin{pmatrix} -b_1\cos\frac{\sqrt{3}}{2}t - b_2\sin\frac{\sqrt{3}}{2}t \\ (a_1-b_1)\cos\frac{\sqrt{3}}{2}t+(a_2-b_2)\sin\frac{\sqrt{3}}{2}t \end{pmatrix},$$

so it follows by an identification of the coefficients of the first row that

$$-\frac{1}{2}a_1 + \frac{\sqrt{3}}{2}a_2 = -b_1, \quad \text{thus} \quad b_1 = \frac{1}{2}a_1 - \frac{\sqrt{3}}{2}a_2,$$

$$-\frac{\sqrt{3}}{2}a_1 - \frac{1}{2}a_2 = -b_2, \quad \text{thus} \quad b_2 = \frac{\sqrt{3}}{2}a_1 + \frac{1}{2}a_2.$$

We shall not calculate the latter two equations from the second row. One may if necessary use them as a control.

Since b_1 and b_2 are uniquely determined by a_1 and a_2, the complete solution of the homogeneous equation with a_1 and a_2 as the arbitrary constants becomes

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a_1 e^{-t/2} \begin{pmatrix} \cos\frac{\sqrt{3}}{2}t \\ \frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t \end{pmatrix} + a_2 e^{-t/2} \begin{pmatrix} \sin\frac{\sqrt{3}}{2}t \\ -\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\sin\frac{\sqrt{3}}{2}t \end{pmatrix}.$$

Remark 2.1 When we compare with the solution of (1), it follows that $a_1 = 2c_1$ og $a_2 = 2c_2$.

Remark 2.2 Since

$$\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t = \cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{3}\right), \qquad -\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\sin\frac{\sqrt{3}}{2}t = \sin\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{3}\right),$$

the complete solution can be written

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a_1 e^{-t/2} \begin{pmatrix} \cos\frac{\sqrt{3}}{2}t \\ \cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{3}\right) \end{pmatrix} + a_2 e^{-t/2} \begin{pmatrix} \sin\frac{\sqrt{3}}{2}t \\ \sin\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{3}\right) \end{pmatrix}.$$

However, this reformulation is not necessary.

3) **Alternatively** we may use the **"fumbling method"**. Expanding the homogeneous system of equations we get

$$\begin{cases} \dfrac{dx_1}{dt} = -x_2, \qquad \text{thus } x_2 = -\dfrac{dx_1}{dt}, \\[2mm] \dfrac{dx_2}{dt} = x_1 - x_2. \end{cases}$$

Here we eliminate x_2 from the latter equation,

$$-\frac{d^2x_1}{dt^2} = x_1 + \frac{dx_1}{dt},$$

thus

$$\frac{d^2x_1}{dt^2} + \frac{dx_1}{dt} + x_1 = 0 \qquad \text{og} \qquad x_2 = -\frac{dx_1}{dt}.$$

The characteristic polynomial R^2+R+1 has the roots $R = -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}$ (the same as the eigenvalues), so the complete solution is

$$x_1(t) = a_1 e^{-t/2}\cos\frac{\sqrt{3}}{2}t + a_2 e^{-t/2}\sin\frac{\sqrt{3}}{2}t.$$

Since $x_2 = -dx_1/dt$, it follows that

$$x_2(t) = a_1 e^{-t/2}\left\{\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t\right\} + a_2 e^{-t/2}\left\{-\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\sin\frac{\sqrt{3}}{2}t\right\},$$

which is seen to be equivalent to the previously found solutions.

The inhomogeneous equation. Even if one should know a fundamental matrix, it cannot be recommended to apply the formal solution formula. This would give us the following difficult expression,

$$\mathbf{\Phi}(t) = e^{-t/2}\begin{pmatrix} \cos\frac{\sqrt{3}}{2}t & \sin\frac{\sqrt{3}}{2}t \\ \frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t & -\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\sin\frac{\sqrt{3}}{2}t \end{pmatrix}.$$

Instead we **guess** a particular solution of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1\cos t + a_2\sin t \\ b_1\cos t + b_2\sin t \end{pmatrix}.$$

Now

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_2 \cos t - a_1 \sin t \\ b_2 \cos t - b_1 \sin t \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -b_1 \cos t - b_2 \sin t \\ (a_1 - b_1)\cos t + (a_2 - b_2)\sin t \end{pmatrix},$$

hence by insertion,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (a_2+b_1)\cos t+(-a_1+b_2)\sin t \\ (b_2-a_1+b_1)\cos t+(-b_1-a_2+b_2)\sin t \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

We get by an identification of the coefficients,

$$a_2 + b_1 = 1, \qquad -a_1 + b_2 = 0,$$

$$b_2 - a_1 + b_1 = 0, \qquad -b_1 - a_2 + b_2 = 1,$$

hence $b_1 = 0$, $a_2 = 1$ and $b_2 = a_1 = 2$.

We get the particular solution

$$\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 2\cos t + \sin t \\ 2\sin t \end{pmatrix}.$$

Finally, the complete solution is obtained by adding all solutions of the homogeneous equation found previously. Since this will give us a mess of formulæ, we shall not produce it here).

Example 2.2 *Find the complete solution of the system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2t \\ t \end{pmatrix}.$$

1) **The eigenvalues** are the roots of the polynomial

$$\begin{vmatrix} -3-\lambda & 4 \\ -2 & 1-\lambda \end{vmatrix} = (\lambda+3)(\lambda-1)+8 = \lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 4,$$

hence the complex eigenvalues are $\lambda = -1 \pm 2i$.

2) The corresponding **complex eigenvectors** are cross vector to anyone of the rows in the matrix

$$\begin{pmatrix} -3-\lambda & 4 \\ -2 & 1-\lambda \end{pmatrix} = \begin{pmatrix} -3+1\mp 2i & 4 \\ -2 & 1+1\mp 2i \end{pmatrix} = \begin{pmatrix} -2\mp 2i & 4 \\ -2 & 2\mp 2i \end{pmatrix}.$$

It follows from the first row, $(-2 \mp 2i, 4) = 2(-\{1 \pm i\}, 2)$ that

$$\mathbf{v}_1 = (2, 1+i)^T \quad \text{for } \lambda_1 = -1 + 2i,$$
$$\mathbf{v}_2 = (2, 1-i)^T \quad \text{for } \lambda_2 = -1 - 2i.$$

3) If $\lambda_1 = a + i\omega = -1 + 2i$, where $a = -1$ and $\omega = 2$, then

$$\mathbf{v}_1 = \alpha + i\beta = \begin{pmatrix} 2 \\ 1+i \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ dvs. } \alpha = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ og } \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then we get the **fundamental matrix**,

$$\begin{aligned}
\mathbf{\Phi}(t) &= \left(\mathrm{Re}\left\{e^{(a+i\omega)t}(\alpha+i\beta)\right\} \ \ \mathrm{Im}\left\{e^{(a+i\omega)t}(\alpha+i\beta)\right\}\right) = e^{at}\cos\omega t(\alpha \ \ \beta) + e^{at}\sin\omega t(-\beta \ \ \alpha) \\
&= e^{-t}\cos 2t \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} + e^{-t}\sin 2t \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2e^{-t}\cos 2t & 2e^{-t}\sin 2t \\ e^{-t}(\cos 2t - \sin 2t) & e^{-t}(\cos 2t + \sin 2t) \end{pmatrix}.
\end{aligned}$$

4) This fundamental matrix looks very complicated, so it does not invite one to apply the solution formula.

Instead we **guess** a **particular solution** of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} at+b \\ ct+d \end{pmatrix} \qquad \text{med} \qquad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}.$$

We get by a rearrangement of the differential equation system that

$$\begin{aligned}
\begin{pmatrix} 2t \\ t \end{pmatrix} &= \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} - \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} at+b \\ ct+d \end{pmatrix} \\
&= \begin{pmatrix} a \\ c \end{pmatrix} - \begin{pmatrix} (-3a+4c)t + (-3b+4d) \\ (-2a+c)t + (-2b+d) \end{pmatrix} = \begin{pmatrix} (3a-4c)t + (a+3b-4d) \\ (2a-c)t + (2b+c-d) \end{pmatrix}.
\end{aligned}$$

Then by an identification of the coefficients,

$$3a - 4c = 2, \quad a + 3b - 4d = 0, \quad 2a - c = 1, \quad 2b + c - d = 0.$$

We get from the first and third equation

$$\begin{cases} 3a - 4c = 2, \\ 2a - c = 1, \end{cases} \qquad \text{that} \qquad \begin{cases} a = 2/5 \\ c = -1/5. \end{cases}$$

Then by a rearrangement and insertion into the second and the fourth equation,

$$\begin{cases} 3b - 4d = -a = -2/5, \\ 2b - d = -c = 1/5, \end{cases}, \qquad \text{hence} \qquad \begin{cases} b = 6/25, \\ d = 7/25. \end{cases}$$

If this is put into our guess, we obtain our particular solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} at+b \\ ct+d \end{pmatrix} = \begin{pmatrix} \frac{2}{5}t + \frac{6}{25} \\ -\frac{1}{5}t + \frac{7}{25} \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 10t+6 \\ -5t+7 \end{pmatrix}.$$

5) It follows from the linearity that the **complete solution** is given by a particular solution to which we add all the solutions of the corresponding homogeneous system,

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \frac{1}{25}\begin{pmatrix} 10t+6 \\ -5t+7 \end{pmatrix} + \mathbf{\Phi}(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \frac{1}{25}\begin{pmatrix} 10t+6 \\ -5t+7 \end{pmatrix} + c_1 e^{-t}\begin{pmatrix} 2\cos t \\ \cos 2t - \sin 2t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\sin 2t \\ \cos 2t + \sin 2t \end{pmatrix},\end{aligned}$$

where c_1 and c_2 are arbitrary constants.

6) **Alternatives**

a) **Real calculations of the solutions of the homogeneous equation**. The eigenvalues $\lambda = -1 \pm 2i$ are complex conjugated, so the structure of the solution of the *homogeneous* equation is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}\begin{pmatrix} a_1\cos 2t + a_2 \sin 2t \\ b_1 \cos 2t + b_2 \sin 2t \end{pmatrix}.$$

We get by a calculation,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}\begin{pmatrix} (-a_1+2a_2)\cos 2t + (-2a_1 - a_2)\sin 2t \\ (-b_1+2b_2)\cos 2t + (-2b_1 - b_2)\sin 2t \end{pmatrix},$$

and

$$\begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}\begin{pmatrix} (-3a_1+4b_1)\cos 2t + (-3a_2+4b_2)\sin 2t \\ (-2a_1+b_1)\cos 2t + (-2a_2+b_2)\sin 2t \end{pmatrix}.$$

When the coefficients are identified, we get

$$\begin{aligned} -a_1 + 2a_2 &= -3a_1 + 4b_1, && \text{dvs. } b_1 = \tfrac{1}{2}a_1 + \tfrac{1}{2}a_2, \\ -2a_1 - a_2 &= -3a_2 + 4b_2, && \text{dvs. } b_2 = -\tfrac{1}{2}a_1 + \tfrac{1}{2}a_2, \\ -b_1 + 2b_2 &= -2a_1 + b_1, && \text{dvs. } a_1 = b_1 - b_2, \\ -2b_1 - b_2 &= -2a_2 + b_2, && \text{dvs. } a_2 = b_1 + b_2. \end{aligned}$$

We see that the four equations are consistent, and that the homogeneous equation has the complete solution

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= e^{-t}\begin{pmatrix} (b_1 - b_2)\cos 2t + (b_1+b_2)\sin 2t \\ b_1\cos 2t + b_2 \sin 2t \end{pmatrix} \\ &= b_1 e^{-t}\begin{pmatrix} \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} + b_2 e^{-t}\begin{pmatrix} -\cos 2t + \sin 2t \\ \sin 2t \end{pmatrix},\end{aligned}$$

corresponding to the fundamental matrix

$$\mathbf{\Phi}_1(t) = e^{-t}\begin{pmatrix} \cos 2t + \sin 2t & -\cos 2t + \sin 2t \\ \cos 2t & \sin 2t \end{pmatrix}.$$

Notice that $\mathbf{\Phi}_1(t) \neq \mathbf{\Phi}(t)$ found previously. However, the two different fundamental matrices are of course equivalent.

b) **Direct calculation of the exponential matrix**. Since $\lambda = -1 \pm 2i$, the trick is to put

$$\mathbf{B} = \mathbf{A} - \ \text{Re}\,\lambda \cdot \mathbf{I} = \begin{pmatrix} -2 & 4 \\ -2 & 2 \end{pmatrix} = 2 \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix},$$

and as $\text{Im}\,\lambda = \pm 2i$, to aim at the cosine and the sine series. We first calculate

$$\mathbf{B}^2 = 2^2 \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} = 2^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -2^2 \mathbf{I},$$

which very conveniently gives

$$\mathbf{B}^{2n} = \left(\mathbf{B}^2\right)^n = (-1)^n \cdot 2^{2n} \cdot \mathbf{I} \quad \text{for } n \in \mathbb{N} \ \ (\text{og for } n = 0).$$

Since $\mathbf{I}$ commutes with everything, we get

$$\begin{aligned} \exp(\mathbf{A}t) &= \exp((\mathbf{A}+\mathbf{I})t - \mathbf{I}t) = e^{-t}\exp(\mathbf{B}t) \\ &= e^{-t} \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{B}^n t^n. \end{aligned}$$

We now divide the investigation into the cases of even and odd indices, and use that

$$\mathbf{B}^{2n} = (-1)^n \cdot 2^{2n}\mathbf{I},$$

so

$$\begin{aligned}
\exp(\mathbf{A}t) &= e^{-t}\sum_{n=0}^{\infty}\frac{1}{(2n)!}\mathbf{B}^{2n}t^{2n} + e^{-t}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\mathbf{B}^{2n}t^{2n+1}\mathbf{B} \\
&= e^{-t}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}2^{2n}t^{2n}\mathbf{I} + e^{-t}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}2^{2n}t^{2n+1}\mathbf{B} \\
&= e^{-t}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2t)^{2n}\mathbf{I} + e^{-t}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2t)^{2n+1}\cdot\frac{1}{2}\mathbf{B} \\
&= e^{-t}\cos 2t\,\mathbf{I} + e^{-t}\sin 2t\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} \cos 2t - \sin 2t & 2\sin 2t \\ -\sin 2t & \cos 2t + \sin 2t \end{pmatrix}.
\end{aligned}$$

We note again that the fundamental matrix is different from both $\mathbf{\Phi}(t)$ and $\mathbf{\Phi}_1(t)$ found previously.

c) **The fumbling method**. We first expand the system,

$$\begin{cases} dx_1/dt = -3x_1 + 4x_2 + 2t, \\ dx_2/dt = -2x_1 + x_2 + t. \end{cases}$$

If we use the first equation to eliminate x_2, it follows that

$$(6)\quad 4x_2 = \frac{dx_1}{dt} + 3x_1 - 2t, \ \text{med}\ \frac{d(4x_2)}{dt} = \frac{d^2x_1}{dt^2} + 3\frac{dx_1}{dt} - 2.$$

Then the latter equation of the system is multiplied by 4,

$$\frac{d(4x_2)}{dt} = -8x_1 + 4x_2 + 4t,$$

and we get by an insertion,

$$\frac{d^2x_1}{dt^2} + 3\frac{x_1}{dt} - 2 = -8x_1 + \left\{\frac{dx_1}{dt} + 3x_1 - 2t\right\} + 4t.$$

Then by a rearrangement,

$$\frac{d^2x_1}{dt^2} + 2\frac{dx_1}{dt} + 5x_1 = 2t + 2.$$

The characteristic equation $R^2 + 2R + 5 = 0$ has the roots $R = -1 \pm 2i$ (the same as the eigenvalues in the other variants).

We **guess** a particular solution of the form

$$x_1 = at + b, \qquad \text{thus}\ \frac{dx_1}{dt} = a\ \text{and}\ \frac{d^2x_1}{dt^2} = 0.$$

Then by insertion,

$$2t+2 = 0+2a+5at+5b = 5at+(2a+5b).$$

When we identify the coefficients, we get

$$\begin{cases} 5a = 2, & \text{dvs. } a = 2/5, \\ 2a+5b = 2, & \text{dvs. } b = \frac{1}{5}(2-2a) = 6/25. \end{cases}$$

Hence,

$$x_1 = \frac{2}{5}t + \frac{6}{25} + c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t.$$

Now,

$$\frac{dx_1}{dt} = \frac{2}{5} + (-c_1+2c_2)e^{-t}\cos 2t + (-2c_1-c_2)e^{-t}\sin 2t,$$

so if we put this into (6), then

$$\begin{aligned} 4x_2 &= \frac{dx_1}{dt} + 3x_1 - 2t \\ &= \frac{2}{5} + (-c_1+2c_2)e^{-t}\cos 2t + (-2c_1-c_2)e^{-t}\sin 2t \\ &\qquad + \frac{6}{5}t + \frac{18}{25} + 3c_1 e^{-t}\cos 2t + 3c_2 e^{-t}\sin 2t - 2t \\ &= -\frac{4}{5}t + \frac{28}{25} + (2c_1+2c_2)e^{-t}\cos 2t + (-2c_1+2c_2)e^{-t}\sin 2t, \end{aligned}$$

whence

$$x_2 = -\frac{1}{5}t + \frac{7}{25} + (\frac{1}{2}c_1 + \frac{1}{2}c_2)e^{-t}\cos 2t + (-\frac{1}{2}c_1 + \frac{1}{2}c_2)e^{-t}\sin 2t.$$

Summing up we get in matrix form

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \frac{2}{5}t + \frac{6}{25} \\ -\frac{1}{5}t + \frac{7}{25} \end{pmatrix} + c_1 e^{-t} \begin{pmatrix} \cos 2t \\ \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin 2t \\ \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t \end{pmatrix} \\ &= \frac{1}{25}\begin{pmatrix} 10t+6 \\ -5t+7 \end{pmatrix} + e^{-t} \begin{pmatrix} \cos 2t & \sin 2t \\ \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t & \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \end{aligned}$$

where c_1 and c_2 are arbitrary constants.

Example 2.3 *Given the linear differential equation system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Find $x_2(t)$ if we assume that

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix}.$$

First solution. The eigenvalue method. The eigenvalues are the roots of the polynomial

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix} = (\lambda-1)(\lambda+1)-1 = \lambda^2 - 2. \qquad \text{dvs. } \lambda = \pm\sqrt{2}.$$

We may e.g. choose an eigenvector corresponding to $\lambda = \sqrt{2}$ as $(1, 1-\sqrt{2})$.

An eigenvector corresponding to $\lambda = -\sqrt{2}$ is e.g. $(1, 1+\sqrt{2})$.

The complete solution of the homogeneous system of equation is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{\sqrt{2}t}\begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} + c_2 e^{-\sqrt{2}t}\begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}.$$

Then we guess a particular solution of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a\cos 2t + b\sin 2t \\ c\cos 2t + d\sin 2t \end{pmatrix}.$$

Now,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2b\cos 2t - 2a\sin 2t \\ 2d\cos 2t - 2c\sin 2t \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a\cos 2t + b\sin 2t \\ c\cos 2t + d\sin 2t \end{pmatrix} = \begin{pmatrix} (a-c)\cos 2t + (b-d)\sin 2t \\ -(a+c)\cos 2t - (b+d)\sin 2t \end{pmatrix},$$

so we can also write the system of equations in the following way,

$$\begin{cases} 2b\cos 2t - 2a\sin 2t = (a-c+2)\cos 2t + (b-d)\sin 2t, \\ 2d\cos 2t - 2c\sin 2t = (-a-c)\cos 2t + (-b-d+1)\sin 2t. \end{cases}$$

When the coefficients are identified we get

$$\begin{array}{ll} 2b = a-c+2, & \text{thus } -a+2b+c = 2, \\ -2a = b-d, & \text{thus } 2a+b-d = 0, \\ 2d = -a-c, & \text{thus } a+c+2d = 0, \\ -2c = -b-d+1, & \text{thus } b-2c+d = 1. \end{array}$$

It follows from the first and the third equation that

$$b+c+d = 1,$$

which together with the fourth equation implies $c = 0$. This reduces the equations to

$$\left\{\begin{array}{r} -a+2b=2, \\ 2a+b-d=0, \\ b+d=1, \end{array}\right. \qquad \text{hence} \quad \left\{\begin{array}{r} -a+2b=2, \\ 2a+2b=1, \\ b+d=1, \end{array}\right.$$

thus

$$a=-\frac{1}{3}, \qquad b=\frac{5}{6}, \qquad c=0, \qquad d=\frac{1}{6}.$$

The complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}\cos 2t+\frac{5}{6}\sin 2t \\ \frac{1}{6}\sin 2t \end{pmatrix} + c_1 e^{\sqrt{2}t}\begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} + c_2 e^{-\sqrt{2}t}\begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}.$$

It follows from the initial conditions that

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} c_1 \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix},$$

so $c_1 = c_2 = 0$, i.e.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}\cos 2t + \frac{5}{6}\sin 2t \\ \frac{1}{6}\sin 2t \end{pmatrix},$$

and then finally,

$$x_2(t) = \frac{1}{6}\sin 2t.$$

Second solution. The "fumbling method". We shall actually only find $x_2(t)$, so it would be reasonable to eliminate $x_1(t)$. First we get from

$$\frac{dx_1}{dt} = x_1 - x_2 + 2\cos 2t, \qquad \frac{dx_2}{dt} = -x_1 - x_2 + \sin 2t,$$

that

$$x_1 = -\frac{dx_1}{dt} - x_2 + \sin 2t,$$

which when put into the first equation gives

$$-\frac{d^2x_2}{dt^2} - \frac{dx_2}{dt} + 2\cos 2t = \frac{dx_2}{dt} - x_2 + 2\cos 2t - x_2 + \sin 2t,$$

hence by a rearrangement,

$$(7) \quad \frac{d^2x_2}{dt^2} - 2x_2 = -\sin 2t, \qquad x_2(0) = 0 \qquad \text{og} \qquad \frac{dx_2}{dt}(0) = \frac{1}{3}.$$

If we **guess** a particular solution of the structure $x_2 = a\cos 2t + b\sin 2t$, we get

$$-6a\cos 2t - 6b\sin 2t = -\sin 2t,$$

hence $a = 0$ and $b = \frac{1}{6}$, and we find the particular solution

$$x_2(t) = \frac{1}{6}\sin 2t.$$

It is seen by inspection that it fulfils the initial conditions, and since the solution is unique, we have solved the problem.

Alternatively the complete solution of (7) is given by

$$x_2(t) = \frac{1}{6}\sin 2t + c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}.$$

It follows from the initial conditions that $c_1 = 0$ and $c_2 = 0$, hence the solution is

$$x_2(t) = \frac{1}{6}\sin 2t.$$

Remark 2.3 In both cases the "fumbling method" is much easy to apply than the eigenvalue method.

Example 2.4 *Find the complete solution of the system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}\sin 2t.$$

The eigenvalues are the roots of the polynomial

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = (\lambda+2)\lambda + 2 = \lambda^2 + 2\lambda + 2 = (\lambda+1)^2 + 1,$$

thus $\lambda = a \pm i\omega = -1 \pm i$ where $a = -1$ and $\omega = 1$.

We first **guess** on a particular solution of the structure

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}\begin{pmatrix} a\cos 2t + b\sin 2t \\ c\cos 2t + d\sin 2t \end{pmatrix}.$$

Since

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}\begin{pmatrix} (-a+2b)\cos 2t - (2a+b)\sin 2t \\ (-c+2d)\cos 2t - (2c+d)\sin 2t \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}\begin{pmatrix} c\cos 2t + d\sin 2t \\ -2(a+c)\cos 2t - 2(b+d)\sin 2t \end{pmatrix},$$

we get from the system of differential equations and a multiplication with e^t that

$$\begin{cases} (-a+2b)\cos 2t - (2a+b)\sin 2t = c\cos 2t + (d+1)\sin 2t, \\ (-c+2d)\cos 2t - (2c+d)\sin 2t = -2(a+c)\cos 2t - (2b+2b+1)\sin 2t. \end{cases}$$

When the coefficients are identified it follows that

$$\begin{array}{ll} -a+b = c, & \text{thus } a-b+c = 0, \\ -(2a+b) = d+1, & \text{thus } -2a-b-d = 1, \\ -c+2d = -2(a+c), & \text{thus } 2a+c+2d = 0, \\ -(2c+d) = -(2b+2d+1), & \text{thus } 2b-2c+d = -1. \end{array}$$

We get from the second and the fourth equation that

$$-2a+b-2c = 0,$$

which together with the first equation gives $b = 0$.

The system of equations is then reduced to

$$\begin{array}{ll} a+c = 0, & \text{thus } c = -a, \\ 2a+d = -1, & \text{thus } 2a+d = -1, \\ 2a+c+2d = 0, & \text{thus } a+2d = 0, \end{array}$$

hence

$$d=\frac{1}{3}, \qquad a=-\frac{2}{3}, \qquad c=\frac{2}{3}, \qquad b=0,$$

and a particular solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{3}e^{-t}\begin{pmatrix} -2\cos 2t \\ 2\cos 2t+\sin 2t \end{pmatrix}.$$

We still miss the complete solution of the corresponding homogeneous system of differential equations,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This can of course be found in many ways.

1) **The eigenvalue method**. We have already found the eigenvalues $\lambda = a \pm i\omega = -1 \pm i$ where $a=-1$ and $\omega = 1$. An eigenvector corresponding to $\lambda = -1+i$ is a cross vector to $(1-i,1)$, e.g.

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} = \alpha + i\beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A fundamental matrix is then given by

$$\begin{aligned}
\mathbf{\Phi}(t) &= \left(\operatorname{Re}\left\{e^{(a+i\omega)t}(\alpha+i\beta)\right\} \;\; \operatorname{Im}\left\{e^{(a+i\omega)t}(\alpha+i\beta)\right\}\right) \\
&= e^{at}\cos\omega t(\alpha \;\; \beta) + e^{at}\sin\omega t(-\beta \;\; \alpha) \\
&= e^{-t}\cos t\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} + e^{-t}\sin t\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} \cos t & \sin t \\ -\cos t - \sin t & \cos t - \sin t \end{pmatrix}.
\end{aligned}$$

The complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{3}e^{-t}\begin{pmatrix} -2\cos 2t \\ 2\cos 2t+\sin 2t \end{pmatrix} + c_1 e^{-t}\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

2) **The exponential matrix**. This is given by the formula (where we from the above have $a=-1$ and $\omega = 1$)

$$\begin{aligned}
\exp(\mathbf{A}t) &= e^{at}\left\{\cos\omega t - \frac{a}{\omega}\sin\omega t\right\}\mathbf{I} + \frac{1}{\omega}e^{at}\sin\omega t\cdot\mathbf{A} \\
&= e^{-t}\{\cos t + \sin t\}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^{-t}\sin t\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} \cos t + \sin t & \sin t \\ -2\sin t & \cos t - \sin t \end{pmatrix},
\end{aligned}$$

hence

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{3}e^{-t}\begin{pmatrix} -2\cos 2t \\ 2\cos 2t+\sin 2t \end{pmatrix} + c_1 e^{-t}\begin{pmatrix} \cos t + \sin t \\ -2\sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

3) **Real structure of the solution.** Since $\lambda = -1 \pm i$, the solution must necessarily have the following structure

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} a_1 \cos t + a_2 \sin t \\ b_1 \cos t + b_2 \sin t \end{pmatrix}.$$

Then by a calculation,

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} (-a_1 + a_2) \cos t - (a_1 + a_2) \sin t \\ (-b_1 + b_2) \cos t - (b_1 + b_2) \sin t \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} b_1 \cos t + b_2 \sin t \\ -2(a_1 + b_1) \cos t - 2(a_2 + b_2) \sin t \end{pmatrix}.$$

When we identify the coefficients we get

$$b_1 = -a_1 + a_2, \qquad b_2 = -a_1 - a_2,$$

and (a little superfluous)

$$-b_1 + b_2 = -2(a_1 + b_1), \qquad 2(a_2 + b_2) = b_1 + b_2.$$

We have thus eliminated b_1 and b_2, hence the complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{3} e^{-t} \begin{pmatrix} -2 \cos 2t \\ 2 \cos 2t + \sin 2t \end{pmatrix} + a_1 e^{-t} \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + a_2 e^{-t} \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}.$$

4) **The "fumbling method"**. The homogeneous system is expanded,

$$\frac{dx_1}{dt} = x_2 \qquad \text{and} \qquad \frac{dx_2}{dt} = -2x_1 - 2x_2.$$

If we put the first equation into the last one, it follows by a rearrangement,

$$\frac{d^2x_1}{dt^2} + 2\frac{dx_1}{dt} + 2x_1 = 0 \quad \text{med } R^2 + 2R + 2 = 0 \text{ for } R = -1 \pm i.$$

Hence

$$x_1 = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

and

$$x_2 = \frac{dx_1}{dt} = c_1 e^{-t}(-\cos t - \sin t) + c_2 e^{-t}(\cos t - \sin t),$$

thus

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}.$$

The complete solution of the inhomogeneous system is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{3}e^{-t}\begin{pmatrix} -2\cos 2t \\ 2\cos 2t + \sin 2t \end{pmatrix} + c_1 e^{-t}\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

Example 2.5 *Consider the system*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}\mathbf{x}(t) + \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \qquad t \geq 0.$$

1) Find the complete solution of the system.

2) Let $\mathbf{x}_0(t)$ be the solution, for which $\mathbf{x}_0(0) = \mathbf{0}$. Find $\mathbf{x}_0(1)$.

1) Clearly, the **eigenvalues** are 1 and 2,

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & -2-\lambda \end{vmatrix} = (\lambda - 1)(\lambda + 2) = 0.$$

The corresponding **eigenvectors** are cross vectors to the first row:
If $\lambda = 1$, then $\mathbf{v}_1 = (1, \lambda - 1) = (1, 0)$.

If $\lambda = -2$, then $\mathbf{v}_2 = (1, \lambda - 1) = (1, -3)$.

The complete solution of the homogeneous equation is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} e^{t} & e^{-2t} \\ 0 & -3e^{-2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The inhomogeneous term is a constant vector. Therefore, we **guess** on a particular solution as a constant vector

$$\tilde{\mathbf{x}}(t)=\begin{pmatrix} a \\ b \end{pmatrix},$$

which gives by insertion,

$$\frac{d\tilde{\mathbf{x}}}{dt}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}+\begin{pmatrix} 1 \\ -4 \end{pmatrix}=\begin{pmatrix} a+b+1 \\ -2b-4 \end{pmatrix},$$

hence $b=-2$ and $a=1$, and we have $\tilde{\mathbf{x}}=(1,2)^T$.

If c_1 and c_2 denote the arbitrary constants, the complete solution is given by

$$\mathbf{x}(t)=\begin{pmatrix} 1 \\ -2 \end{pmatrix}+\begin{pmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Alternatively we may apply the **"fumbling method"**. We expand the system,

$$\begin{cases} dx_1/dt=x_1+x_2+1, & \text{thus } dx_1/dt-x_1=x_2+1, \\ \\ dx_2/dt=-2x_1-4, & \text{thus } dx_2/dt+2x_2=-4. \end{cases}$$

Clearly, the solution of the latter equation is

$$x_2=-2+c_2e^{-2t}.$$

When this is put into the first equation, we get

$$\frac{dx_1}{dt}-x_1=-1+c_2e^{-2t},$$

hence

$$x_1=1+c_2e^t\int e^{-t}e^{-2t}\,dt+c_1e^t=1+c_1e^t-\frac{1}{3}c_2e^{-2t},$$

and summing up,

$$\mathbf{x}(t)=\begin{pmatrix} 1 \\ -2 \end{pmatrix}+\begin{pmatrix} e^t & -\frac{1}{3}e^{-2t} \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

2) When we put $t=0$, we get

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}+\begin{pmatrix} c_1+c_2 \\ -3c_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

thus $c_2=-\frac{2}{3}$ and $c_1=-\frac{1}{3}$, so

$$\mathbf{x}_0(t)=\begin{pmatrix} 1 \\ -2 \end{pmatrix}-\frac{1}{3}\begin{pmatrix} e^t \\ 0 \end{pmatrix}-\frac{2}{3}\begin{pmatrix} e^{-2t} \\ -3e^{-3t} \end{pmatrix}.$$

Then finally,

$$\mathbf{x}_0(1)=\begin{pmatrix} 1 \\ -2 \end{pmatrix}-\frac{1}{3}\begin{pmatrix} e \\ 0 \end{pmatrix}-\frac{2}{3}\begin{pmatrix} e^{-2} \\ -3e^{-2} \end{pmatrix}=\frac{1}{e^2}\begin{pmatrix} e^2-\frac{1}{3}e^3-\frac{2}{3} \\ \\ -2e^2+2 \end{pmatrix}.$$

3 Examples of applications in Physics

Example 3.1 *Consider a physical system consisting of two coupled oscillators. We assume that there is no damper in the system. The three spring constants are k_1, k and k_2, and m_1 and m_2 denote the masses of each of the two particles. At equilibrium we assume that the spring forces are 0. It can be proved by Newton's second law that the system can be described by the following system of differential equations,*

$$\frac{d^2x_1}{dt^2}+\frac{k_1+k}{m_1}x_1=\frac{k}{m_1}x_2 \quad \textit{and} \quad \frac{d^2x_2}{dt^2}+\frac{k_2+k}{m_2}x_2=\frac{k}{m_2}x_1.$$

Put $m_1=m_2=1$, $k=\frac{3}{10}$, $k_1=\frac{8}{5}$, $k_2=\frac{4}{5}$, and assume that

$$x_1(0)=3\cdot 10^{-2}, \qquad x_1'(0)=0,$$

$$x_2(0)=3\cdot 10^{-2}, \qquad x_2'(0)=0.$$

Find $x_1(t)$ and $x_2(t)$ as solutions of a differential equation of fourth order.

By using the selected values of m_1, m_2 k, k_1 and k_2, we get

$$\frac{d^2x_1}{dt^2}=-\frac{k_1+k}{m_1}x_1+\frac{k}{m_1}x_2=-\frac{19}{10}x_1+\frac{3}{10}x_2,$$

$$\frac{d^2x_2}{dt^2}=\frac{k}{m_2}x_1-\frac{k+k_2}{m_2}x_2=\frac{3}{10}x_1-\frac{11}{10}x_2.$$

We immediately get three different methods of solution.

1) **The traditional eigenvalue method**. If we put

$$y_1=x_1, \quad y_2=\frac{dx_1}{dt}, \quad y_3=x_2 \quad \text{og} \quad y_4=\frac{dx_2}{dt},$$

then we get the homogeneous system

$$\frac{d}{dt}\begin{pmatrix} y_1\\ y_2\\ y_3\\ y_4\end{pmatrix}=\begin{pmatrix} y_2\\ -\frac{19}{10}y_1+\frac{3}{10}y_3\\ y_4\\ \frac{3}{10}y_1-\frac{11}{10}y_3\end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0\\ -\frac{19}{10} & 0 & \frac{3}{10} & 0\\ 0 & 0 & 0 & 1\\ \frac{3}{10} & 0 & -\frac{11}{10} & 0\end{pmatrix}\begin{pmatrix} y_1\\ y_2\\ y_3\\ y_4\end{pmatrix}.$$

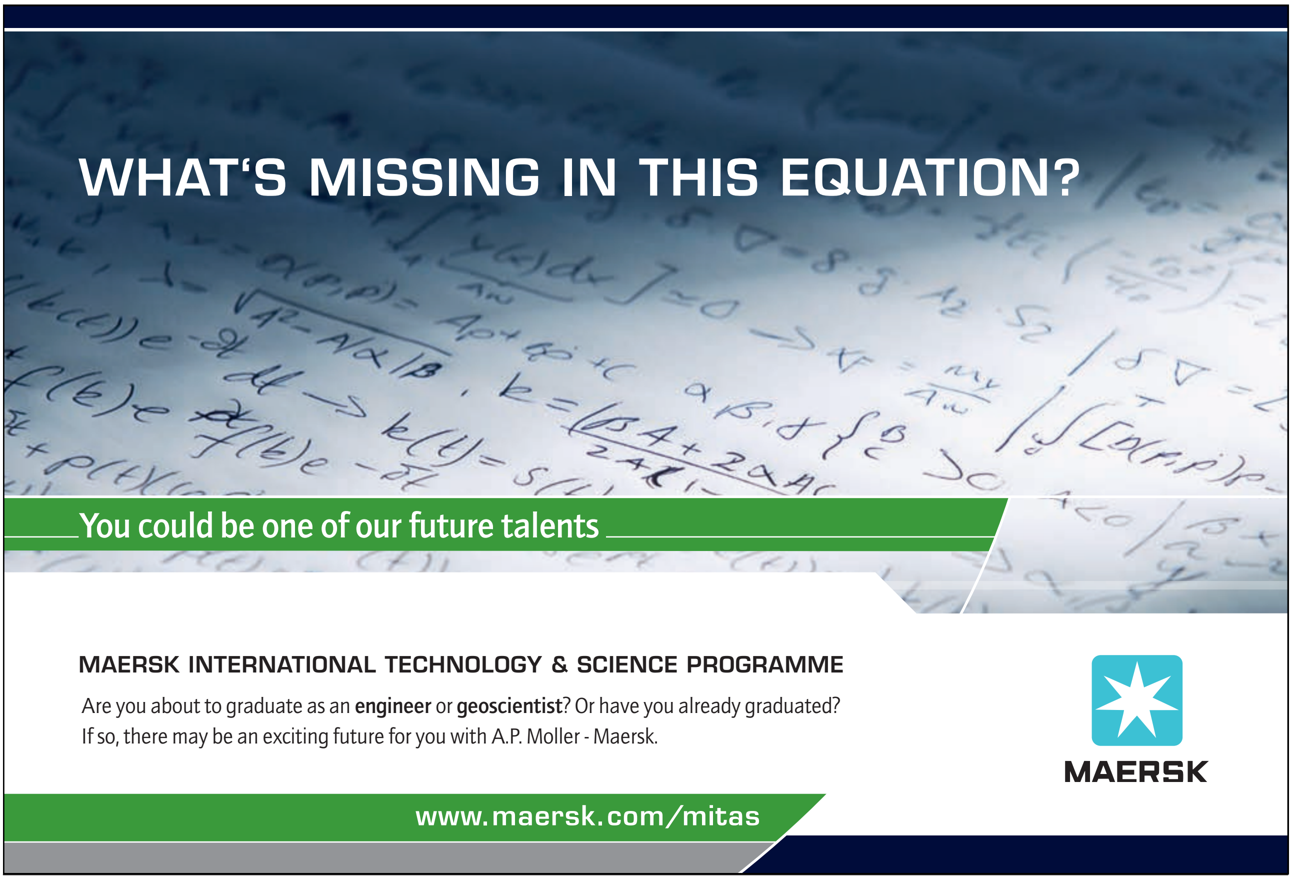
The eigenvalues are the roots of the polynomial

$$\begin{aligned}
\begin{vmatrix} -\lambda & 1 & 0 & 0\\ -\frac{19}{10} & -\lambda & \frac{3}{10} & 0\\ 0 & 0 & -\lambda & 1\\ \frac{3}{10} & 0 & -\frac{11}{10} & -\lambda\end{vmatrix}
&= -\lambda\begin{vmatrix} -\lambda & \frac{3}{10} & 0\\ 0 & -\lambda & 1\\ 0 & -\frac{11}{10} & -\lambda\end{vmatrix}-\begin{vmatrix} -\frac{19}{10} & \frac{3}{10} & 0\\ 0 & -\lambda & 1\\ \frac{3}{10} & -\frac{11}{10} & -\lambda\end{vmatrix}\\
&= \lambda^2\begin{vmatrix} -\lambda & 1\\ -\frac{11}{10} & -\lambda\end{vmatrix}+\lambda\begin{vmatrix} -\frac{19}{10} & 0\\ \frac{3}{10} & -\lambda\end{vmatrix}+\begin{vmatrix} -\frac{19}{10} & \frac{3}{10}\\ \frac{3}{10} & -\frac{11}{10}\end{vmatrix}\\
&= \lambda^2\left(\lambda^2+\frac{11}{10}\right)+\frac{19}{10}\lambda^2+\frac{1}{100}(19\cdot 11-9)\\
&= \lambda^4+3\lambda^2+2=(\lambda^2+1)(\lambda^2+2),
\end{aligned}$$

thus the eigenvalues are $\lambda = \pm i$ and $\lambda = \pm\sqrt{2}i$.

It is here fairly difficult to find the complex eigenvectors, so we note instead that the structure of the solution must be of the form

$$
\begin{aligned}
y_1 &= x_1(t) = a_1 \cos t + a_2 \sin t + a_3 \cos\sqrt{2}t + a_4 \sin\sqrt{2}t, \\
y_2 &= \frac{dx_1}{dt} = -a_1 \sin t + a_2 \cos t - \sqrt{2}a_3 \sin\sqrt{2}t + \sqrt{2}a_4 \cos\sqrt{2}t, \\
y_3 &= x_2(t) = b_1 \cos t + b_2 \sin t + b_3 \cos\sqrt{2}t + b_4 \sin\sqrt{2}t, \\
y_4 &= \frac{dx_2}{dt} = -b_1 \sin t + b_2 \cos t - \sqrt{2}b_3 \sin\sqrt{2}t + \sqrt{2}b_4 \cos\sqrt{2}t.
\end{aligned}
$$



Since

$$\frac{d^2x_1}{dt^2} = -a_1\cos t - a_2\sin t - 2a_3\cos\sqrt{2}t - 2a_4\sin\sqrt{2}t,$$

$$\frac{d^2x_2}{dt^2} = -b_1\cos t - b_2\sin t - 2b_3\cos\sqrt{2}t - 2b_4\sin\sqrt{2}t,$$

and

$$\begin{aligned}-\frac{19}{10}x_1+\frac{3}{10}x_2 &= \left(-\frac{19}{10}a_1+\frac{3}{10}b_1\right)\cos t+\left(-\frac{19}{10}a_2+\frac{3}{10}b_2\right)\sin t \\ &\quad +\left(-\frac{19}{10}a_3+\frac{3}{10}b_3\right)\cos\sqrt{2}t+\left(-\frac{19}{10}a_4+\frac{3}{10}\right)\sin\sqrt{2}t,\end{aligned}$$

and

$$\begin{aligned}\frac{3}{10}x_1-\frac{11}{10}x_2 &= \left(\frac{3}{10}a_1-\frac{11}{10}b_1\right)\cos t+\left(\frac{3}{10}a_2-\frac{11}{10}b_2\right)\sin t \\ &\quad +\left(\frac{3}{10}a_3-\frac{11}{10}b_3\right)\cos\sqrt{2}t+\left(\frac{3}{10}a_4-\frac{11}{10}b_4\right)\sin\sqrt{2}t,\end{aligned}$$

we get by an identification of the coefficients that

$$-a_1 = -\frac{19}{10}a_1+\frac{3}{10}b_1, \qquad \text{thus } b_1 = 3a_1,$$

$$-a_2 = -\frac{19}{10}a_2+\frac{3}{10}b_2, \qquad \text{thus } b_2 = 3a_2,$$

$$-2a_3 = -\frac{19}{10}a_3+\frac{3}{10}b_3, \qquad \text{thus } b_3 = -\frac{1}{3}a_3,$$

$$-2a_4 = -\frac{19}{10}a_4+\frac{3}{10}b_4, \qquad \text{thus } b_4 = -\frac{1}{3}a_4.$$

This gives us the general solution

$$x_1(t)=a_1\cos t+a_2\sin t+a_3\cos\sqrt{2}t+a_4\sin\sqrt{2}t,$$

$$x_2(t)=3a_1\cos t+3a_2\sin t-\frac{1}{3}a_3\cos\cos\sqrt{2}t-\frac{1}{3}a_4\sin\sqrt{2}t.$$

Since

$$x_1'(t)=-a_1\sin t+a_2\cos t-a_3\sqrt{2}\sin\sqrt{2}t+a_4\sqrt{2}\cos\sqrt{2}t,$$

$$x_2'(t)=-3a_1\sin t+3a_2\cos t+a_3\frac{\sqrt{2}}{3}\sin\sqrt{2}t-a_4\frac{\sqrt{2}}{3}\cos\sqrt{2}t,$$

it follows from the initial conditions that

$$x_1(0) = \frac{3}{100} = a_1+a_3, \qquad x_2(0) = \frac{3}{100} = 3a_1-\frac{1}{3}a_3,$$

$$x_1'(0) = 0 = a_2+a_4\sqrt{2}, \qquad x_2'(0) = 0 = 3a_2-a_4\frac{\sqrt{2}}{3}.$$

We conclude from the latter two equations that $a_2 = a_4 = 0$. Then

$$a_3 = 9a_1 - \frac{9}{100}$$

implies that

$$a_1 = \frac{3}{100} - \left(9a_1 - \frac{9}{100}\right), \qquad \text{dvs. } a_1 = \frac{12}{1000} = \frac{3}{250}$$

and

$$a_3 = \frac{3}{100} - a_1 = \frac{3}{100} - \frac{3}{250} = \frac{15-6}{500} = \frac{9}{500}.$$

The wanted solution is then given by

$$\begin{cases} x_1 = \dfrac{3}{250}\cos t + \dfrac{9}{500}\cos(\sqrt{2}t), \\[2mm] x_2 = \dfrac{9}{250}\cos t - \dfrac{3}{500}\cos(\sqrt{2}t). \end{cases}$$

2) **Alternatively** the system can be written

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{19}{10} & \frac{3}{10} \\ \frac{3}{10} & -\frac{11}{10} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The eigenvalues are the roots of the polynomial

$$\begin{vmatrix} -\frac{19}{10} - \lambda & \frac{3}{10} \\ \frac{3}{10} & -\frac{11}{10} - \lambda \end{vmatrix} = \left(\lambda + \frac{19}{10}\right)\left(\lambda + \frac{11}{10}\right) - \frac{9}{100} = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2),$$

thus either $\lambda = -1$ or $\lambda = -2$.

An eigenvector of the eigenvalue $\lambda = -1$ is e.g. $(1,3)$, corresponding to the differential equation

$$\frac{d^2}{dt^2}(x_1 + 3x_2) = -(x_1 + 3x_2),$$

the complete solution of which is

(8) $x_1 + 3x_2 = a_1 \cos t + a_2 \sin t.$

An eigenvector of the eigenvalue $\lambda = -2$ is e.g. $\left(1, -\frac{1}{3}\right)$, corresponding to the differential equation

$$\frac{d^2}{dt^2}\left(x_1 - \frac{1}{3}x_2\right) = -2\left(x_1 - \frac{1}{3}x_2\right),$$

the complete solution of which is

(9) $x_1 - \frac{1}{3}x_2 = b_1 \cos(\sqrt{2}t) + b_2 \sin(\sqrt{2}t).$

By solving (8) and (9) we get the complete solution

$$x_1 = \frac{1}{10}a_1 \cos t + \frac{1}{10}a_2 \sin t + \frac{9}{10}b_1 \cos \sqrt{2}t + \frac{9}{10}b_2 \sin \sqrt{2}t,$$

$$x_2 = \frac{3}{10}a_1 \cos t + \frac{3}{10}a_2 \sin t - \frac{3}{10}b_1 \cos \sqrt{2}t - \frac{3}{10}b_2 \sin \sqrt{2}t.$$

Then it follows from the initial conditions that

$$\begin{cases} x_1(0) = \dfrac{1}{10}a_1 + \dfrac{9}{10}b_1 = \dfrac{3}{100}, \\[2ex] x_2(0) = \dfrac{3}{10}a_1 - \dfrac{3}{10}b_1 = \dfrac{3}{100}, \end{cases} \qquad \text{dvs.} \qquad \begin{cases} a_1 + 9b_1 = \dfrac{3}{10}, \\[2ex] a_1 - b_1 = \dfrac{1}{10}, \end{cases}$$

hence $b_1 = \dfrac{1}{50}$ and $a_1 = \dfrac{3}{25}$. It follows from

$$x_1'(0) = \frac{1}{10}a_2 + \frac{9\sqrt{2}}{10}b_2 = 0 \quad \text{and} \quad x_2'(0) = \frac{3}{10}a_2 - \frac{3\sqrt{2}}{10}b_2 = 0,$$

that $a_2 = b_2 = 0$.

The wanted solution is

$$\begin{cases} x_1 = \dfrac{3}{250}\cos t + \dfrac{9}{500}\cos(\sqrt{2}t), \\[2ex] x_2 = \dfrac{9}{250}\cos t - \dfrac{3}{500}\cos(\sqrt{2}t). \end{cases}$$

3) **The "fumbling method"**. If we eliminate x_2 by

$$x_2 = \frac{m_1}{k}\frac{d^2x_1}{dt^2} + \frac{k_1+k}{k}x_1,$$

then

$$\frac{m_1}{k}\frac{d^4x_1}{dt^4} + \frac{k_1+k}{k}\frac{d^2x_1}{dt^2} + \frac{k_2+k}{k}\frac{m_1}{m_2}\frac{d^2x_1}{dt^2} + \frac{(k_1+k)(k_1+k_2)}{km_2}x_1 = \frac{k}{m_2}x_1,$$

hence by a rearrangement,

$$\frac{m_1}{k}\frac{d^4x_1}{dt^4} + \frac{k_1+k_2+2k}{k}\frac{d^2x_1}{dt^2} + \frac{(k_1+k_2)k+k_1k_2}{km_2}x_1 = 0.$$

When we multiply by k and insert the chosen values of k, m_i and k_j, we get

$$0 = \frac{d^4x_1}{dt^4} + \left(\frac{3}{5} + \frac{8}{5} + \frac{4}{5}\right)\frac{d^2x_1}{dt^2} + \left(\frac{3}{10}\cdot\frac{12}{5} + \frac{32}{25}\right)x_1 = \frac{d^4x_1}{dt^4} + 3\frac{d^2x_1}{dt^2} + 2x_1.$$

The characteristic polynomial $R^4+3R^2+2 = (R^2+1)(R^2+2)$ has the roots $R = \pm i$ and $R = \pm\sqrt{2}i$, thus

$$x_1 = c_1 \cos t + c_2 \sin t + c_3 \cos(\sqrt{2}t) + c_4 \sin(\sqrt{2}t),$$

and whence

$$\begin{aligned} x_2 &= \frac{10}{3}\frac{d^2x_1}{dt^2} + \frac{10}{3}\left(\frac{3}{10}+\frac{8}{5}\right)x_1 = \frac{10}{3}\frac{d^2x_1}{dt^2} + \frac{19}{3}x_1 \\ &= 3c_1\cos t + 3c_2\sin t - \frac{1}{3}c_3\cos(\sqrt{2}t) - \frac{1}{3}c_4\sin(\sqrt{2}t). \end{aligned}$$

It follows from the initial conditions that

$$x_1(0) = c_1 + c_3 = \frac{3}{100}, \qquad x_2(0) = 3c_1 - \frac{1}{3}c_3 = \frac{3}{100},$$

$$x_1'(0) = c_2 + \sqrt{2}c_4 = 0, \qquad x_1'(0) = 3c_2 - \frac{\sqrt{2}}{2}c_4 = 0.$$

We immediately get

$$c_2 = c_4 = 0, \qquad \text{and } c_1 = \frac{3}{250} \text{ og } c_3 = \frac{9}{500}.$$

The wanted solution is

$$\begin{cases} x_1 = \dfrac{3}{250}\cos t + \dfrac{9}{500}\cos(\sqrt{2}t), \\[2mm] x_2 = \dfrac{9}{250}\cos t - \dfrac{3}{500}\cos(\sqrt{2}t). \end{cases}$$

Example 3.2 *For small oscillations (small swings Θ and φ) it is possible to show that the model of the double pendulum can be described by the equations*

$$2\ell\frac{d^2\Theta}{dt^2} + \ell\frac{d^2\varphi}{dt^2} + 2g\Theta = 0,$$

$$\ell\frac{d^2\Theta}{dt^2} + \ell\frac{d^2\varphi}{dt^2} + g\varphi = 0.$$

Find the eigenfrequencies and the complete solution.

If we solve with respect to (Θ,φ), we get the system

$$\begin{pmatrix} \Theta \\ \varphi \end{pmatrix} = \begin{pmatrix} -\ell/g & -\ell/(2g) \\ -\ell/g & -\ell/g \end{pmatrix} \frac{d^2}{dt^2}\begin{pmatrix} \Theta \\ \varphi \end{pmatrix} = \mathbf{A}\frac{d^2}{dt^2}\begin{pmatrix} \Theta \\ \varphi \end{pmatrix}.$$

The eigenvalues satisfy the equations

$$\begin{vmatrix} -(\ell/g)-\lambda & -\ell/(2g) \\ -\ell/g & -(\ell/g)-\lambda \end{vmatrix} = \left(\lambda + \frac{\ell}{g}\right)^2 - \frac{1}{2}\left(\frac{\ell}{g}\right)^2 = 0,$$

thus $\lambda = \left(-1 \pm \dfrac{\sqrt{2}}{2}\right)\dfrac{\ell}{g}$ and a corresponding eigenvector is e.g. $(1, \mp\sqrt{2})$.

Since $\mathbf{A}^{-1}$ has the same eigenvectors as $\mathbf{A}$, and the eigenvalues $\dfrac{1}{\lambda}$, we derive the two differential equations of second order

$$\frac{d^2}{dt^2}(\Theta - \sqrt{2}\varphi) = -(2+\sqrt{2})\frac{\ell}{g}(\Theta - \sqrt{2}\varphi),$$

$$\frac{d^2}{dt^2}(\Theta + \sqrt{2}\varphi) = -(2-\sqrt{2})\frac{\ell}{g}(\Theta + \sqrt{2}\varphi),$$

hence

$$\Theta - \sqrt{2}\varphi = 2a_1\cos\left(\sqrt{2+\sqrt{2}}\sqrt{\frac{\ell}{g}}t\right) + 2a_2\sin\left(\sqrt{2+\sqrt{2}}\sqrt{\frac{\ell}{g}}t\right),$$

$$\Theta+\sqrt{2}\varphi = 2b_1\cos\left(\sqrt{2-\sqrt{2}}\sqrt{\frac{\ell}{g}}t\right)+2b_2\sin\left(\sqrt{2-\sqrt{2}}\sqrt{\frac{\ell}{g}}t\right).$$

Finally, we get

$$\begin{aligned}\Theta &= a_1\cos\left(\sqrt{\frac{(2+\sqrt{2})\ell}{g}}t\right)+a_2\sin\left(\sqrt{\frac{(2+\sqrt{2})\ell}{g}}t\right)\\ &\quad +b_1\cos\left(\sqrt{\frac{(2-\sqrt{2})\ell}{g}}t\right)+b_2\sin\left(\sqrt{\frac{(2-\sqrt{2})\ell}{g}}t\right),\end{aligned}$$

and

$$\begin{aligned}\varphi &= -\frac{a_1}{\sqrt{2}}\cos\left(\sqrt{\frac{(2+\sqrt{2})\ell}{g}}t\right)-\frac{a_2}{\sqrt{2}}\sin\left(\sqrt{\frac{(2+\sqrt{2})\ell}{g}}t\right)\\ &\quad +\frac{b_1}{\sqrt{2}}\cos\left(\sqrt{\frac{(2-\sqrt{2})\ell}{g}}t\right)+\frac{b_2}{\sqrt{2}}\sin\left(\sqrt{\frac{(2-\sqrt{2})\ell}{g}}t\right).\end{aligned}$$

Example 3.3 *Two electric conductors are coupled inductively. If i_1 and i_2 denote the current intensities of the conductors, then the induced forces $M\dfrac{di_2}{dt}$ and $M\dfrac{di_1}{dt}$, (where M is a constant) in each of the conductors, resp.. Then the differential equations of i_1 and i_2 are given by*

$$L_1\frac{d^2i_1}{dt^2}+R_1\frac{di_1}{dt}+\frac{1}{C_1}i_1+M\frac{d^2i_2}{dt^2}=0,$$

$$M\frac{d^2i_1}{dt^2}+L_2\frac{d^2i_2}{dt^2}+R_2\frac{di_2}{dt}+\frac{1}{C_2}i_2=0,$$

where L, R and C are the induction coefficient, the resistance and the capacity, resp..

1) Find the complete solution

2) Check the cases

a) $M=0$,

b) $R_1=R_2=0$, and $n_1=\dfrac{1}{\sqrt{L_1C_1}}=n_2=\dfrac{1}{\sqrt{L_2C_2}}$.

1) If we put $x_1=i_1$, $x_2=i_2$, $x_3=\dfrac{di_1}{dt}$ and $x_4=\dfrac{di_2}{dt}$, then

$$L_1\frac{dx_3}{dt}+R_1x_3+\frac{1}{C_1}x_1+M\frac{dx_4}{dt}=0,$$

$$M\frac{dx_3}{dt}+L_2\frac{dx_4}{dt}+R_2x_4+\frac{1}{C_2}x_2=0,$$

thus by a rearrangement,

$$L_1\frac{dx_3}{dt}+M\frac{dx_4}{dt}=-\frac{1}{C_1}x_1-R_1x_3,$$

$$M\frac{dx_3}{dt}+L_2\frac{dx_4}{dt}=-\frac{1}{C_2}x_2-R_2x_4.$$

If $L_1L_2\neq M^2$, then

$$\frac{dx_3}{dt}=-\frac{L_2x_1}{(L_1L_2-M^2)C_1}+\frac{Mx_2}{(L_1L_2-M^2)C_2}-\frac{R_1L_2x_3}{L_1L_2-M^2}+\frac{MR_2x_4}{L_1L_2-M^2},$$

$$\frac{dx_4}{dt}=\frac{Mx_1}{(L_1L_2-M^2)C_1}-\frac{L_1x_2}{(L_1L_2-M^2)C_2}+\frac{R_1Mx_3}{L_1L_2-M^2}-\frac{L_1R_2x_4}{L_1L_2-M^2}.$$

Hence in the form of a matrix,

$$\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x},$$

so if we put $a=L_1L_2-M^2$, then

$$\mathbf{A}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-\dfrac{L_2}{aC_1}&\dfrac{M}{aC_2}&-\dfrac{R_1L_2}{a}&\dfrac{MR_2}{a}\\\dfrac{M}{aC_1}&-\dfrac{L_2}{aC_2}&\dfrac{R_1M}{a}&-\dfrac{L_1R_2}{a}\end{pmatrix}.$$

In principle it is possible to find the eigenvalues and the eigenfunctions of this system. In practice, however, it is very difficult, so we stop here.

4 Stability

Example 4.1 *Check the stability of the following system*

$$\frac{\delta \mathbf{x}}{dt} = \begin{pmatrix} 1 & 7 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mathbf{u}(t).$$

The eigenvalues are the roots of the polynomial

$$\begin{vmatrix} 1-\lambda & 7 \\ 3 & -2-\lambda \end{vmatrix} = (\lambda-1)(\lambda+2) - 21 = \lambda^2 + \lambda - 23.$$

This polynomial has a negative coefficient, hence the system is unstable.

Example 4.2 *Check the stability of the system*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & 1 & 0 \\ -5 & -1 & 1 \\ -7 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \mathbf{u}(t).$$

The eigenvalues are the roots of the polynomial

$$\begin{aligned} \begin{vmatrix} -1-\lambda & 1 & 0 \\ -5 & -1-\lambda & 1 \\ -7 & 0 & 1-\lambda \end{vmatrix} &= (\lambda+1)^2(1-\lambda) - 7 + 5(1-\lambda) \\ &= \lambda^3 + \lambda^2 + 4\lambda + 1. \end{aligned}$$

Here $a_1 = 1 > 0$, $a_2 = 4 > 0$, $a_3 = 1 > 0$, and

$$\begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 3 > 0,$$

so all roots have a negative real part, and the system is asymptotically stable.

Example 4.3 *Check the stability of the system*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \mathbf{u}(t).$$

The eigenvalues are the roots of the polynomial

$$\begin{aligned} \begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -1-\lambda & -1 & -1 \\ -1 & 0 & -1-\lambda & -1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} &= -(\lambda+1) \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & -1-\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} \\ &= (\lambda+1)\{\lambda^2(-1-\lambda) - 1 - \lambda\} = (\lambda+1)^2(\lambda^2+1). \end{aligned}$$

The roots are $\lambda = -1$ (double root) and $\lambda = \pm i$.

The system is stable, but not asymptotically stable.

Example 4.4 *Check the stability of the system*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mathbf{u}(t).$$

The eigenvalues are the roots of the polynomial

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-1)+1 = \lambda^2,$$

so $\lambda = 0$ is a *double root*. It is not possible at this stage to conclude anything about the stability, so we must necessarily solve the system.

It follows from **Cayley-Hamilton's theorem** (cf. Linear Algebra) that $\mathbf{A}^2 = \mathbf{0}$, hence the series of the exponential matrix is reduced to

$$\exp(\mathbf{A}t) = \mathbf{I} + t\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} t & -t \\ t & -t \end{pmatrix} = \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}.$$

The complete solution of the homogeneous equation is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1+t \\ t \end{pmatrix} + c_2 \begin{pmatrix} -t \\ 1-t \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + t \begin{pmatrix} c_1 - c_2 \\ c_1 - c_2 \end{pmatrix}.$$

If $c_1 \neq c_2$, then the absolute value of this solution tends to infinity, so we conclude that the system is unstable.

Example 4.5 *Check the stability of the system,*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The eigenvalues are the roots of the polynomial

$$\begin{vmatrix} 1-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} = (\lambda-1)(\lambda-2) - 2 = \lambda^2 - 3\lambda = \lambda(\lambda-3),$$

hence $\lambda = 0$ and $\lambda = 3 > 0$, and we conclude that the system is unstable.

Example 4.6 *Check the stability of the system,*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \cos t \\ \cos 2t \\ \sin t \end{pmatrix}.$$

The eigenvalues are the roots of the polynomial

$$\begin{vmatrix} -2-\lambda & -1 & 0 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = -(\lambda+1) \begin{vmatrix} \lambda+2 & 1 \\ 1 & \lambda+1 \end{vmatrix} = -(\lambda+1)\{\lambda^2 + 3\lambda + 1\}.$$

Since all coefficients in the splitting into factors have the same sign, every root must have a negative real part, and we conclude that the system is asymptotically stable.

Example 4.7 *Check the stability of the system,*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \mathbf{u}(t).$$

The eigenvalues are the roots of the polynomial

$$\begin{aligned} \begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & -1-\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} &= (-1)^3 \begin{vmatrix} \lambda-1 & 1 & 0 \\ 0 & \lambda+1 & 1 \\ -1 & 1 & \lambda \end{vmatrix} = -\{\lambda(\lambda^2-1)-1-(\lambda-1)\} \\ &= -\{\lambda^3-\lambda-\lambda\} = -\lambda(\lambda^2-2) = \lambda(\lambda-\sqrt{2})(\lambda+\sqrt{2}). \end{aligned}$$

It follows immediately that the system is unstable.

Example 4.8 *Find all numbers a, for which the linear system*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & a & 1+a^2 \\ 0 & -a & 0 & a \\ -a & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \mathbf{u}(t)$$

is asymptotically stable.

The eigenvalues are the roots of the polynomial

$$\begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -1-\lambda & a & 1+a^2 \\ 0 & -a & -\lambda & a \\ -a & 1 & 0 & -1-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -1-\lambda & a & 1+a^2 \\ -a & -\lambda & a \\ 1 & 0 & -1-\lambda \end{vmatrix} + a \begin{vmatrix} -1-\lambda & a \\ -a & -\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -1-\lambda & a & a^2-\lambda \\ -a & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} + a\{\lambda^2+\lambda+a^2\}$$

$$= -\lambda \begin{vmatrix} a & a^2-\lambda \\ -\lambda & 0 \end{vmatrix} + \lambda^2 \begin{vmatrix} -1-\lambda & a \\ -a & -\lambda \end{vmatrix} + a\{\lambda^2+\lambda+a^2\}$$

$$= -\lambda(-\lambda^2+a^2\lambda)+\lambda^2(\lambda^2+\lambda+a^2)+a(\lambda^2+\lambda+a^2) = \lambda^4+2\lambda^3+a\lambda^2+a\lambda+a^3,$$

hence

$$a_1 = 2, \quad a_2 = a, \quad a_3 = a, \quad a_4 = a^3, \quad \text{and } n = 4.$$

We get from Routh-Hurwitz's criterion the conditions $D_1 = a_1 = 2 > 0$,

$$D_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} = \begin{vmatrix} 2 & a \\ 1 & a \end{vmatrix} = a > 0,$$

$$\begin{aligned} D_3 &= \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 2 & a & 0 \\ 1 & a & a^3 \\ 0 & 2 & a \end{vmatrix} = a \begin{vmatrix} 2 & a & 0 \\ 1 & a & a^2 \\ 0 & 2 & 1 \end{vmatrix} \\ &= a(2a-a-4a^2) = a^2(1-4a) > 0. \end{aligned}$$

It follows that the condition for asymptotically stability is that $0 < a < \dfrac{1}{4}$.

Example 4.9 *Let $(x, h)^T$ denote a state vector (where h denotes the velocity of M, defined below). A servo system, which is used to keep the (right hand side of M) in a constant position x_0 independently of the external force $f(t)$ on M, can then be described by the state equations,*

$$\frac{d}{dt}\begin{pmatrix} x \\ h \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \dfrac{Ke_0}{MrR} - \dfrac{k}{M} & -\dfrac{K^2}{r^2RM} \end{pmatrix}\begin{pmatrix} x \\ h \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{f(t)}{M} - \dfrac{Ke_0x_0}{RrM} \end{pmatrix}.$$

Here the spring has the equilibrium length 0, and the error of the position governs the dependent generator.

1) *Find the characteristic polynomial of the system, and the values of e_0, for which the system is stable.*

2) *Assume that $f(t) = F$ is constant and that the system is stable. Find $x_1 = \lim_{t\to\infty} x(t)$. Is $x_1 = x_0$?*

3) *Assume that $f(t)$ is arbitrary for $t \in [0, t_0[$, while $f(t)$ is 0 for $t > t_0$. Find $\lim_{t\to\infty} x(t)$.*

1) The characteristic polynomial is

$$P(\lambda) = \begin{vmatrix} -\lambda & 1 \\ \dfrac{Ke_0}{MrR} - \dfrac{k}{M} & -\lambda - \dfrac{K^2}{r^2RM} \end{vmatrix} = \lambda^2 + \frac{K^2}{r^2RM}\lambda + \frac{k}{M} - \frac{Ke_0}{MrR}.$$

The system is asymptotically stable, when

$$0 < \frac{k}{M} - \frac{Ke_0}{MrR} = \frac{K}{MrR}\left\{\frac{krR}{K} - e_0\right\},$$

hence when

$$0 < e_0 < \frac{krR}{K}.$$

2) We shall find a particular solution $(x_1, h_1)^T$. If $f(t) = F$ is a constant, we guess on a constant vector (x_1, h_1). It follows from the former equation that

$$\frac{dx_1}{dt} = 0 = 0 \cdot x_1 + h_1 + 0, \qquad \text{dvs. } h_1 = 0.$$

By insertion into the latter equation we obtain

$$\frac{dh_1}{dt} = 0 = \left\{\frac{Ke_0}{MrR} - \frac{k}{M}\right\}x_1 + \frac{F}{M} - \frac{Ke_0x_0}{RrM},$$

thus

$$\frac{1}{MrR}\{Ke_0 - krR\}x_1 = \frac{1}{RrM}\{Ke_0x_0 - FRr\},$$

and hence

$$x_1 = \frac{Ke_0x_0 - FRr}{Ke_0 - krR}.$$

The solutions of the homogeneous equation die out when $e < \dfrac{krR}{K}$, so the expression is equal to $\lim_{t\to\infty} x(t)$. (Note that the denominator is < 0). It follows that this expression is only equal to x_0, if $F = kx_0$.

3) If the process is initiated after t_0, it follows that we can choose $F = 0$. By insertion of this into the result of (3), we get by the assumption $e_0 < \dfrac{krR}{K}$ that

$$\lim_{t \to \infty} x(t) = \frac{Ke_0x_0}{Ke_0 - krR} < 0.$$

Example 4.10 *Check if the solutions of the differential equation*

$$y''' + 4y'' + 4y = 0$$

are stable.

The characteristic polynomial is

$$P(\lambda) = \lambda^3 + 4\lambda^2 + 4 = \lambda^3 + 4\lambda^2 + 0 \cdot \lambda + 4.$$

The coefficient of λ is 0, hence the system is not asymptotically stable.

By means of e.g. a pocket calculator we find the approximate roots

$$\lambda = 0,112085 \pm 0,966627i \qquad \text{eller} \qquad \lambda = -4,22417.$$

Since there are roots with a positive real part, the system is unstable.

Alternatively we introduce the disturbance

$$P_\varepsilon(\lambda) = \lambda^3 + 4\lambda^2 + \varepsilon\lambda + 4.$$

Then all roots have a negative real part, if and only if $\varepsilon > 0$ and

$$\begin{vmatrix} 4 & 4 \\ 1 & \varepsilon \end{vmatrix} = 4(\varepsilon - 1) > 0 \qquad \text{(Routh-Hurwitz's criterion)},$$

thus if and only if $\varepsilon > 1$. Since we here let $\varepsilon \to 0$, we again conclude that the system is unstable.

Alternatively the equation has a real root < 0 and two complex conjugated roots $x \pm iy$. When we put $\lambda = x + iy$, $y \neq 0$, then

$$0 = (x+iy)^3 + 4(x+iy)^2 + 4 = \{x^3 - 3xy^2 + 4x^2 - 4y^2 + 4\} + i \cdot y(3x^2 - y^2 + 8x).$$

Since in particular the imaginary part is 0, we must necessarily have that $y^2 = 3x^2 + 8x$, which when put into the real part gives the *necessary condition*

$$0 = -8x^3 - 32x^2 - 32x + 4.$$

Since we have both positive and negative coefficients, we must have a real and positive root, so the system is unstable.

Example 4.11 *It is well-known that a rigid body can be in a permanent rotation around any of its principal axes (through a fixed point of the body). However, the rotation around the axis of the "middle" moment of inertia is not stable. Apply Euler's equations and small variations of the velocity of the angle to prove this.*
Euler's equations are

$$I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\omega_2\omega_3 = M_1,$$

$$I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\omega_1\omega_3 = M_2,$$

$$I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\omega_1\omega_2 = M_3.$$

Assume that $M_1 = M_2 = M_3 = 0$ and $\omega_1 = \omega_0 + \xi_1$, $\omega_2 = \xi_2$, $\omega_3 = \xi_3$, where ξ_ν are small variations and ω_0 is a constant (hence one consider a rotation around the first principal axis and small disturbances). By insertion into Euler's equations, follows by a linearization we obtain a system of first order for ξ_ν, the stability of which should be checked.

Putting $M_1 = M_2 = M_3 = 0$ and $\omega_1 = \omega_0 + \xi_1$, $\omega_2 = \xi_2$, $\omega_3 = \xi_3$ into Euler's equations we get by linearizations (this implies that we assume that the ξ_ν-erne are so small that we can neglect terms of higher order) that

$$0 = I_1 \frac{d\xi_1}{dt} + (I_3 - I_2)\xi_2\xi_3 \approx I_1 \frac{d\xi_1}{dt},$$

$$0 = I_2 \frac{d\xi_2}{dt} + (I_1 - I_3)(\omega_0 + \xi_1)\xi_3 \approx I_2 \frac{d\xi_2}{dt} + (I_1 - I_3)\omega_0 \xi_3,$$
$$0 = I_3 \frac{d\xi_3}{dt} + (I_2 - I_1)(\omega_0 + \xi_1)\xi_2 \approx I_3 \frac{d\xi_3}{dt} + (I_2 - I_1)\omega_0 \xi_2.$$

This linearization is written in matrix form

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dfrac{I_1 - I_3}{I_2}\omega_0 \\ 0 & -\dfrac{I_2 - I_3}{I_3}\omega_0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}.$$

The characteristic polynomial is

$$-\lambda \begin{vmatrix} -\lambda & -\dfrac{I_1 - I_3}{I_2}\omega_0 \\ -\dfrac{I_2 - I_3}{I_3}\omega_0 & -\lambda \end{vmatrix} = -\lambda \left\{ \lambda^2 - \frac{(I_1 - I_3)(I_2 - I_1)}{I_2 I_3}\omega_0^2 \right\}.$$

Since ξ_1 is a constant, we obtain stability (though not asymptotically stability), when

$$\frac{(I_1 - I_3)(I_2 - I_1)}{I_2 I_3} < 0,$$

thus when

$$(I_1 - I_2)(I_1 - I_3) > 0.$$

For fixed I_2 and I_3 this is only possible when I_1 does *not* lie between I_2 and I_3. Therefore, if I_1 is the "middle" moment of inertia, then we have unstability.

Remark 4.1 It follows easily from Euler's original equations that

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \geq 0 \qquad \text{is a constant.}$$

In fact,

$$\begin{aligned} \frac{d}{dt}\{I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\} &= 2I_1\omega_1 \frac{\delta\omega_1}{dt} + 2I_2\omega_2 \frac{d\omega_2}{dt} + 2I_3\omega_3 \frac{d\omega_3}{dt} \\ &= 2\omega_1\omega_2\omega_3\{I_1(I_2 - I_3) + I_2(I_3 - I_1) + I_3(I_1 - I_2)\} = 0. \end{aligned}$$

Example 4.12 *Consider the linear system*

$$(10)\quad \frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 2t, \qquad t \in \mathbb{R}.$$

1) *Find the complete solution of (10) by first finding a solution of the inhomogeneous equation, and then find the complete solution of the homogeneous equation.*

2) *Then prove that (10) has periodical solutions which unlike the external force* $\cos 2t$ *does not have the period* π.

3) *Is it possible for a stable and linear system for a given external periodical force to have a periodical solution of a different period than the external force?*

1) We guess a particular solution of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \cos 2t + a_2 \sin 2t \\ b_1 \cos 2t + b_2 \sin 2t \end{pmatrix}.$$

Then

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2a_2 \cos 2t - 2a_1 \sin 2t \\ 2b_2 \cos 2t - 2b_1 \sin 2t \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -b_1 \cos 2t - b_2 \sin 2t \\ a_1 \cos 2t + a_2 \sin 2t \end{pmatrix}.$$

By insertion into the equation and an identification of the coefficients we get

$$\begin{cases} 2a_2 + b_1 = 1, \\ 2a_1 = b_2, \\ 2b_2 = a_1, \\ 2b_1 = -a_2, \end{cases} \qquad \text{hvoraf} \begin{cases} a_1 = b_2 = 0, \\ a_2 = \frac{2}{3}, \\ b_1 = -\frac{1}{3}. \end{cases}$$

A particular solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2\sin 2t \\ -\cos 2t \end{pmatrix}.$$

It follows immediately that the eigenvalues are $\lambda = \pm i$ and that $(\cos t, \sin t)$ and $(\sin t, -\cos t)$ are linearly independent solutions of the homogeneous equation. Hence the complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2\sin 2t \\ -\cos 2t \end{pmatrix} + c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

2) It is obvious that if $c_1 \neq 0$ or $c_2 \neq 0$, then every solution is periodical with period 2π.

3) The answer if affirmative, because we have above produced an example. Notice that this system is clearly stable, (though it is not asymptotically stable).

Example 4.13 *Given the linear system of differential equations*

$$\frac{dx_1}{dt} = x_1 - 8x_2,$$
$$\frac{dx_2}{dt} = -x_1 + 3x_2.$$

1) Find a fundamental matrix of the system.

2) Is the system asymptotically stable?

3) Find the solution $\mathbf{x}(t)$ *of the system, for which* $\mathbf{x}(0) = (6, 0)^T$.

1) Here there are lots of variants, of which we demonstrate two of them.

a) **The eigenvalue method**. The system is on matrix form,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -8 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The eigenvalues are the solutions of the equation

$$\begin{vmatrix} 1-\lambda & -8 \\ -1 & 3-\lambda \end{vmatrix} = (\lambda-1)(\lambda-3)-8 = \lambda^2-4\lambda-5 = (\lambda-5)(\lambda+1) = 0,$$

hence $\lambda_1 = 5$ and $\lambda_2 = -1$. The eigenvectors are cross vectors of $(-1, 3-\lambda)$.
If $\lambda_1 = 5$, then e.g. $\mathbf{v}_1 = (2, -1)$.
If $\lambda_2 = -1$, then e.g. $\mathbf{v}_2 = (4, 1)$.
The complete solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{5t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{5t} & 4e^{-t} \\ -e^{5t} & e^{-t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

hence a fundamental matrix is given by

$$\mathbf{\Phi}(t) = \begin{pmatrix} 2e^{5t} & 4e^{-t} \\ -e^{5t} & e^{-t} \end{pmatrix}.$$

b) **The fumbling method**. We eliminate x_1 by

$$x_1 = -\frac{x_2}{dt} + 3x_2.$$

Then

$$\frac{dx_1}{dt} = -\frac{d^2x_2}{dt^2} + 3\frac{dx_2}{dt} = x_1 - 8x_2 = -\frac{dx_2}{dt} + 3x_2 - 8x_2 = -\frac{dx_2}{dt} - 5x_2,$$

hence by a rearrangement,

$$\frac{d^2x_2}{dt^2} - 4\frac{x_2}{dt} - 5x_2 = 0 \quad \text{med } R^2-4R-5 = (R-5)(R+1).$$

The complete solution is

$$x_2 = c_1 e^{5t} + c_2 e^{-t} \quad \text{med} \quad \frac{dx_2}{dt} = 5c_1 e^{5t} - c_2 e^{-t},$$

thus

$$x_1 = -\frac{dx_2}{dt} + 3x_2 = -5c_1 e^{5t} + c_2 e^{-t} + 3c_1 e^{5t} + 3c_2 e^{-t} = -2c_1 e^{5t} + 4c_2 e^{-t}.$$

Summing up we get

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2c_1 e^{5t} + 4c_2 e^{-t} \\ c_1 e^{5t} + c_2 e^{-t} \end{pmatrix} = \begin{pmatrix} -2e^{5t} & 4e^{-t} \\ e^{5t} & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and a fundamental matrix is given by

$$\mathbf{\Phi}_1(t) = \begin{pmatrix} -2e^{5t} & 4e^{-t} \\ e^{5t} & e^{-t} \end{pmatrix}.$$

2) The system has a positive eigenvalue, hence the system is unstable.

3) We shall find (c_1, c_2) of the system of equations,

$$\begin{pmatrix} 6 \\ 0 \end{pmatrix} = \mathbf{\Phi}(0)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

thus

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 6 \\ 0 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

and the wanted solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2e^{5t} + 4e^{-t} \\ -e^{5t} + e^{-t} \end{pmatrix}.$$

Example 4.14 *Given the linear system of differential equations*

$$\begin{cases} \dfrac{dx_1}{dt} = 5x_1 + ax_2, \\ \dfrac{dx_2}{dt} = 2x_1 + bx_2, \end{cases} \qquad a,\, b \in \mathbb{R}.$$

1) Find a relation, which a and b must satisfy, if the system is asymptotically stable.

2) Find for $a = -4$ and $b = -1$ a fundamental matrix for the system.

3) Find $e^{\mathbf{A}t}$ for $\mathbf{A} = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$.

1) The characteristic polynomial is

$$\begin{vmatrix} 5-\lambda & a \\ 2 & b-\lambda \end{vmatrix} = (\lambda - b)(\lambda - 5) - 2a = \lambda^2 - (5+b)\lambda + (5b - 2a).$$

It follows from **Routh-Hurwitz's criterion** that the system is asymptotically stable, if and only if

$$-(5+b) > 0 \qquad \text{and} \qquad 5b - 2a > 0,$$

hence if and only if $\dfrac{2}{5}\,a < b < -5$.

2) When $a = -4$ and $b = -1$ the characteristic polynomial becomes

$$\lambda^2 - (5-1)\lambda - 5 + 8 = \lambda^2 - 4\lambda + 3 = (\lambda - 2)^2 - 1 = (\lambda - 1)(\lambda - 3),$$

thus the roots are $\lambda = 1$ and $\lambda = 3$.

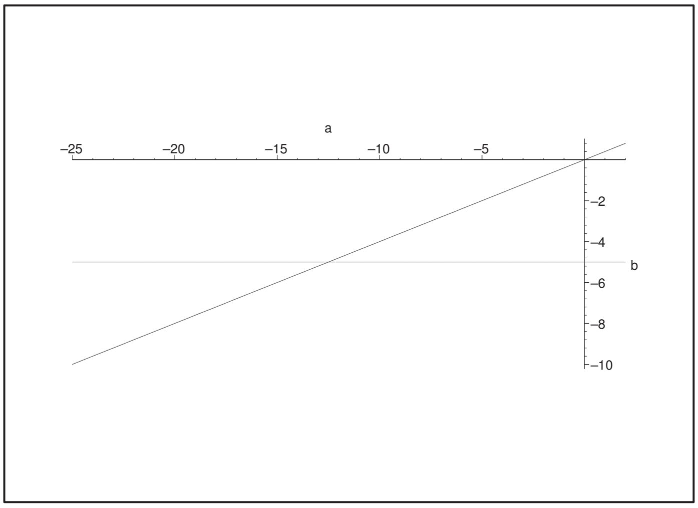
Since the matrix is $\mathbf{A}$, given in (3), an eigenvector corresponding to an eigenvalue λ is a cross vector of $(5 - \lambda, -4)$.

If $\lambda_1 = 1$, then $\mathbf{v}_1 = (1, 1)^T$.

If $\lambda_2 = 3$, then $\mathbf{v}_2 = (2, 1)^T$.

A fundamental matrix is

$$\mathbf{\Phi}(t) = \left(e^t\mathbf{v}_1, e^{3t}\mathbf{v}_2\right) = \begin{pmatrix} e^t & 2e^{3t} \\ e^t & e^{3t} \end{pmatrix}.$$



3) If we instead use the fundamental matrix $\mathbf{\Phi}(t)$, found in (2), we get

$$\mathbf{\Phi}(0)=\begin{pmatrix}1&2\\1&1\end{pmatrix}\qquad \text{med } \mathbf{\Phi}(0)^{-1}=-\begin{pmatrix}1&-2\\-1&1\end{pmatrix}=\begin{pmatrix}-1&2\\1&-1\end{pmatrix}.$$

Then

$$\exp(\mathbf{A}t)=\mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}=\begin{pmatrix}e^t&2e^{3t}\\e^t&e^{3t}\end{pmatrix}\begin{pmatrix}-1&2\\1&1\end{pmatrix}=\begin{pmatrix}-e^t+2e^{3t}&2e^t-2e^{3t}\\-e^t+e^{3t}&2e^t-e^{3t}\end{pmatrix}.$$

Alternatively,

$$\begin{aligned}\exp(\mathbf{A}t)&=\frac{-\lambda_2e^{\lambda_1t}+\lambda_1e^{\lambda_2t}}{\lambda_1-\lambda_2}\,\mathbf{I}+\frac{e^{\lambda_1t}-e^{\lambda_2t}}{\lambda_1-\lambda_2}\,\mathbf{A}=-\frac{1}{2}\left\{-3e^t+e^{3t}\right\}\mathbf{I}-\frac{1}{2}\left\{e^t-e^{3t}\right\}\mathbf{A}\\&=\frac{1}{2}\begin{pmatrix}3e^t-e^{3t}&0\\0&3e^t-e^{3t}\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-5e^t+5e^{3t}&4e^t-4e^{3t}\\-2e^t+2e^{3t}&e^t-e^{3t}\end{pmatrix}\\&=\begin{pmatrix}-e^t+2e^{3t}&2e^t-2e^{3t}\\-e^t+e^{3t}&2e^t-e^{3t}\end{pmatrix}.\end{aligned}$$

Example 4.15 *Find a relationship between the real parameters a, b, such that the linear system*

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1&a\\1&b\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

is asymptotically stable.

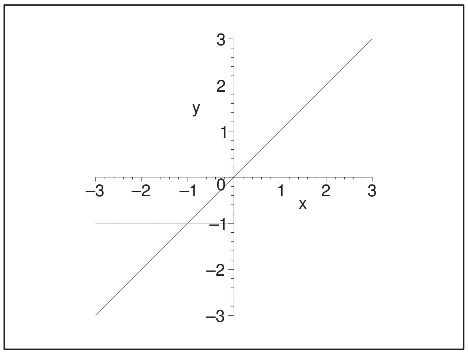
The characteristic polynomial is

$$\begin{vmatrix}1-\lambda&a\\1&b-\lambda\end{vmatrix}=(\lambda-1)(\lambda-b)-a=\lambda^2-(b+1)\lambda+(b-a).$$

It follows from Routh-Hurwitz's criterion that the system is asymptotically stable, if and only if

$$-(b+1) > 0 \qquad \text{og} \qquad b-a>0,$$

thus if $a<b<-1$.



Example 4.16 *Let*

$$\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -11 & -7 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} -3 & -1 & -2 \\ 0 & -1 & 0 \\ 4 & 0 & -3 \end{pmatrix}.$$

1) Check if the linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x}$$

is asymptotically stable.

2) Prove, e.g. by means of Routh-Hurwitz's criterium, that the linear system

$$\frac{d\mathbf{y}}{dt} = \mathbf{B}\,\mathbf{y}$$

is asymptotically stable.

1) The characteristic polynomial for $\mathbf{A}$ is given by

$$\begin{vmatrix} 6-\lambda & 4 \\ -11 & -7-\lambda \end{vmatrix} = (\lambda-6)(\lambda+7)+44 = \lambda^2+\lambda+2 = \left(\lambda+\frac{1}{2}\right)^2 + \frac{7}{4}.$$

We have here two variants:

a) Since all coefficients of the characteristic polynomial are positive, it follows immediately from Routh-Hurwitz's criterion that the system is asymptotically stable.

b) Since the roots $\lambda = -\frac{1}{2} \pm i\sqrt{\frac{7}{4}}$ all have negative real part, the system is asymptotically stable.

2) If we expand the determinant after the second row, we get the characteristic polynomial for $\mathbf{B}$,

$$\begin{aligned}
\begin{vmatrix} -3-\lambda & -1 & -2 \\ 0 & -1-\lambda & 0 \\ 4 & 0 & -3-\lambda \end{vmatrix} &= -(\lambda+1)\begin{vmatrix} -3-\lambda & -2 \\ 4 & -3-\lambda \end{vmatrix} \\
&= -(\lambda+1)\{(\lambda+3)^2+8\} = (\lambda+1)\{\lambda^2+6\lambda+17\} \qquad (11) \\
&= -\{\lambda^3+7\lambda^2+23\lambda+17\} = -\{\lambda^3+a_1\lambda^2+a_2\lambda+a_3\}, \qquad (12)
\end{aligned}$$

hence $a_1 = 7$, $a_2 = 23$ and $a_3 = 17$.

We have again two variants:

a) It follows from (11) that the roots are

$$-1, \qquad -3+i2\sqrt{2}, \qquad -3-i2\sqrt{2}.$$

They have all a negative real part, hence the system is asymptotically stable.

b) The conditions of Routh-Hurwitz's criterion are [cf. (12)]

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0.$$

The first three relations are clearly satisfied. Finally,

$$\begin{vmatrix} 1_1 & a_3 \\ 1 & a_2 \end{vmatrix} = \begin{vmatrix} 7 & 17 \\ 1 & 23 \end{vmatrix} = 161 - 17 = 144 > 0.$$

Then by **Routh-Hurwitz's criterion** the linear system is asymptotically stable.

Example 4.17 *Check if the linear system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

is asymptotically stable.

The characteristic polynomial is

$$-p(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & -1 & -2-\lambda \end{vmatrix} = -\lambda^2(\lambda+2) - 1 - \lambda = -\{\lambda^3 + 2\lambda^2 + \lambda + 1\},$$

thus

$$p(\lambda) = \lambda^3 + 2\lambda^2 + \lambda + 1 = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3.$$

Now, $a_1 = 2 > 0$, $a_2 = 1 > 0$ and $a_3 = 1 > 0$, and

$$\begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 > 0,$$

so it follows from **Routh-Hurwitz's criterion** that the system is asymptotically stable.

Remark 4.2 By using a pocket calculator it is seen that the roots are approximatively

$$\lambda_{1,2} = -0,122561 \pm i \cdot 0,744862, \quad \lambda_3 = -1,75488.$$

Example 4.18 *Check if the linear system*

$$\frac{d}{dt} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is asymptotically stable?

The eigenvalues of the matrix satisfy

$$0 = \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 9 \qquad [= \lambda^2 - 4\lambda - 5],$$

so the eigenvalues are

$$\lambda = 2 \pm 3 = \begin{cases} 5, \\ -1. \end{cases}$$

We see that there exists a positive eigenvalue, hence the system is not asymptotically stable.

Remark 4.3 We mention for completeness that the complete solution is

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

5 Transfer functions

Example 5.1 *Let* $\mathbf{A}$ *denote the matrix*

$$\mathbf{A} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

1) Find the transfer functions $H_1(s)$ *og* $H_2(s)$ *for the systems*

a)

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t), \quad t \in \mathbb{R}, \qquad y(t) = (1,1)\mathbf{x}(t),$$

b)

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t) + \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} u(t), \quad t \in \mathbb{R}, \quad y(t) = (1,1)\mathbf{x}(t).$$

2) Find the stationary solution of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t) + \begin{pmatrix} 2\cos t - \frac{1}{2}\cos 2t \\ \frac{1}{2}\cos 2t \end{pmatrix}, \quad t \in \mathbb{R}, \qquad y(t) = (1,1)\mathbf{x}(t),$$

where we first prove that the system is stable.

The characteristic polynomial is

$$P(\lambda) = \begin{vmatrix} -\lambda - \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\lambda - \frac{1}{2} \end{vmatrix} = \left(\lambda + \frac{3}{2}\right)\left(\lambda + \frac{1}{2}\right) + \frac{1}{4} = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2,$$

so $\lambda = -1$ is an eigenvalue of the multiplicity 2. The system is in particular asymptotically stable, and we have proved the first part of (2).

1) a) Here $\mathbf{c}^T = (1,1)$, $\mathbf{b} = (1,0)^T$ and $d = 0$, and

$$s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s + \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & s + \frac{1}{2} \end{pmatrix} \qquad \text{where } \det(s\mathbf{I} - \mathbf{A}) = (s+1)^2,$$

Then

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)^2} \begin{pmatrix} s + \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & s + \frac{3}{2} \end{pmatrix}, \qquad s \neq -1,$$

hence

$$\begin{aligned} H_1(s) &= \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = (1,1)\frac{1}{(s+1)^2}\begin{pmatrix} s + \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & s + \frac{3}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{(s+1)^2}(1,1)\begin{pmatrix} s + \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \frac{s}{(s+1)^2}, \qquad s \neq -1. \end{aligned}$$

b) Since $\mathbf{c}^T = (1,1)$ and $\mathbf{b} = (-\frac{1}{2}, \frac{1}{2})^T$ and $d = 0$, and since $(s\mathbf{I} - \mathbf{A})^{-1}$ was computed in (a), we get

$$\begin{aligned} H_2(s) &= \frac{1}{(s+1)^2}\,(1,1)\begin{pmatrix} s+\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & s+\frac{3}{2} \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2(s+1)^2}\,(s, s+2)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{(s+1)^2}. \end{aligned}$$

2) We have already in the beginning proved that the system is stable. Now,

$$\begin{pmatrix} 2\cos t - \frac{1}{2}\cos 2t \\ \frac{1}{2}\cos 2t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 2\cos t + \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \cos 2t,$$

and

$$2\cos t = 2\,\mathrm{Re}\left(e^{it}\right) \qquad \text{and} \qquad \cos 2t = \mathrm{Re}\left(e^{2it}\right),$$

so it follows by applying (1a), (1b) and the linearity that the stationary solution is

$$\begin{aligned} y(t) &= 2\,\mathrm{Re}\{H_1(i)e^{it}\} + \mathrm{Re}\{H_2(2i)e^{2it}\} \\ &= 2\,\mathrm{Re}\left\{\frac{i}{(1+i)^2}\,e^{it}\right\} + \mathrm{Re}\left\{\frac{1}{(1+2i)^2}\,e^{2it}\right\} \\ &= 2\,\mathrm{Re}\left\{\frac{i}{2i}e^{it}\right\} + \mathrm{Re}\left\{\frac{(1-2i)^2}{25}\,e^{2it}\right\} \\ &= \cos t + \frac{1}{25}\,\mathrm{Re}\{(-3-4i)(\cos 2t + i\sin 2t)\} \\ &= \cos t - \frac{3}{25}\cos 2t + \frac{4}{25}\sin 2t. \end{aligned}$$

Example 5.2 *Consider the linear system*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{x}(t) + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u(t), \qquad y(t) = (1,1)\mathbf{x}(t).$$

1) Prove that the system is stable.

2) Find the stationary solution, when $u(t) = 4\cos t$.

1) The characteristic polynomial

$$P(\lambda) = \begin{vmatrix} -1-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 + 2$$

has the roots $\lambda = -1 \pm i\sqrt{2}$, which both lie in the left hand half plane, so the system si asymptotically stable.

2) First find the transfer function. Since

$$\mathbf{c}^T = (1,1), \qquad \mathbf{b} = (-1,1)^T, \qquad d = 0,$$

and

$$s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s+1 & 1 \\ -2 & s+1 \end{pmatrix} \qquad \text{where } \det(s\mathbf{I}-\mathbf{A}) = (s+1)^2+2,$$

and

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{(s+1)^2+2}\begin{pmatrix} s+1 & -1 \\ 2 & s+1 \end{pmatrix},$$

the transfer function is

$$\begin{aligned} H(s) &= (1,1)\,\frac{1}{(s+1)^2+2}\begin{pmatrix} s+1 & -1 \\ 2 & s+1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \frac{(1,1)}{(s+1)^2+2}\begin{pmatrix} -s-2 \\ s-1 \end{pmatrix} = -\frac{3}{(s+1)^2+2}. \end{aligned}$$

Since $4\cos t = \operatorname{Re}\{4e^{it}\}$, the stationary solution is

$$\begin{aligned} y(t) &= \operatorname{Re}\left\{H(i)4e^{it}\right\} = \operatorname{Re}\left\{-\frac{3}{2+2i}\cdot 4e^{it}\right\} = \operatorname{Re}\left\{\frac{-6}{1+i}\,e^{it}\right\} \\ &= \operatorname{Re}\{-3(1-i)e^{it}\} = \operatorname{Re}\{(-3+3i)(\cos t + i\sin t)\} \\ &= -3\cos t - 3\sin t = -3\sqrt{2}\sin\left(t+\frac{\pi}{4}\right), \end{aligned}$$

which clearly is periodical with period 2π.

Example 5.3 *Consider the linear system of differential equations of first order*

$$\text{(13)} \quad \frac{d\mathbf{x}}{dt} = a\mathbf{x}(t) + \mathbf{u}(t), \qquad t \in \mathbb{R}.$$

1) Find the values of the constant a, for which there for every periodical exterior force $\mathbf{u}(t)$ of period T exists precisely one periodical solution of (13) with period T.

2) Find a value of the constant a and a periodical exterior force $\mathbf{u}(t)$ of period T, such that

a) (13) does not have any periodical solutions of period T.

b) (13) has infinitely many periodical solutions of period T.

Using the coordinates, (13) is written

$$\frac{dx_j}{dt} = ax_j(t) + u_j(t), \qquad t \in \mathbb{R}.$$

Hence we may assume that the dimension is 1, so (13) is reduced to

$$\frac{dx}{dt} = ax(t) + u(t),$$

the complete solution of which is

$$x(t) = ce^{at} + e^{at}\int e^{-at}u(t)\,dt.$$

1) If $a \notin \left\{ \frac{2\pi n}{T} i \mid x \in \mathbb{Z} \right\}$, then there is precisely one periodical solution for every periodical exterior force.

2) Choose e.g. $a = 0$. (Any $a = \frac{2\pi n}{T} i$ can actually be chosen).

   a) If $u(t) = \left| \sin\left(\frac{2\pi}{T} t \right) \right|$, then $u(t)$ has the period T. Since $u(t) \geq 0$, it follows that $x(t) = \int u(t)\, dt$ is *not* periodical.

   b) If $u(t) = \sin\left(\frac{2\pi}{T} t \right)$, then every solution

$$x(t) = c - \frac{T}{2\pi} \cos\left(\frac{2\pi}{T} t \right)$$

   is periodic.

Example 5.4 *Given a stable linear system with the external force $u(t)$ and the given transfer function*

$$H(s) = \frac{s+2}{s^2+2s+4}.$$

Find the stationary solution, when

$$(1)\quad u(t) = 2\cos\left(2t+\frac{\pi}{4}\right), \qquad (2)\quad u(t) = -\sin 4t.$$

1) Since

$$2\cos\left(2t+\frac{\pi}{4}\right) = 2\text{Re}\left\{e^{2it}\cdot\exp\left(i\frac{\pi}{4}\right)\right\} = \sqrt{2}\,\text{Re}\{(1+i)e^{2it}\},$$

and

$$H(2i) = \frac{2i+2}{-4+4i+4} = \frac{1+i}{2i},$$

we obtain the real stationary solution

$$y(t) = \sqrt{2}\,\text{Re}\left\{\frac{1+i}{2i}(1+i)e^{2it}\right\} = \sqrt{2}\,\text{Re}\{e^{2it}\} = \sqrt{2}\cos 2t.$$

2) Since

$$-\sin 4t = -\,\text{Im}\{e^{4it}\},$$

and

$$H(4i) = \frac{4i+2}{-16+8i+4} = -\frac{2(1+2i)}{4(3-2i)}\cdot\frac{3+2i}{3+2i} = -\frac{1}{26}(-1+8i),$$

we obtain the real stationary solution

$$y(t) = \frac{1}{26}\,\text{Im}\{(-1+8i)(\cos 4t + i\sin 4t)\} = \frac{1}{26}\{8\cos 4t - \sin 4t\}.$$

Example 5.5 *A linear system of first order with one external force $u(t)$ and the response $y(t)$ has the given transfer function*

$$H(s) = \frac{1}{1+s}.$$

1) Prove that the system is stable.

2) Find the amplitude and phase for the stationary solution, when

$$(a)\ \ u(t) = \cos t, \quad (b)\ \ u(t) = 2\cos 2t,$$

$$(c)\ \ u(t) = -\cos t, \quad (d)\ \ u(t) = \sin 2t.$$

1) The transfer function is given by

$$H(s) = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d.$$

This expression is *not* defined, if and only if s is an eigenvalue for $\mathbf{A}$. In the given case we see that $H(s)$ is not defined for $s = -1 < 0$, which lies in the left hand half plane, so the system is asymptotical stable.

2) a) Since $u(t) = \cos t = \text{ Re } e^{it}$, and $H(i) = \dfrac{1}{1+i} = \dfrac{1}{2}(1-i)$, we get the real stationary solution with a phase shift

$$\begin{aligned} y(t) &= \operatorname{Re}\{H(i)e^{it}\} = \frac{1}{2}\operatorname{Re}\{(1-i)e^{it}\} = \frac{1}{2}(\cos t + \sin t) \\ &= \frac{1}{\sqrt{2}}\sin\left(t + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\cos\left(t - \frac{\pi}{4}\right). \end{aligned}$$

b) Since $u(t) = 2\cos 2t = 2\operatorname{Re}\, e^{2it}$, and

$$H(2i) = \frac{1}{1+2i} = \frac{1}{5}(1-2i),$$

we get the real stationary solution

$$\begin{aligned} y(t) &= = \operatorname{Re}\{H(2i)e^{2it}\} = \frac{1}{5}\{(1-2i)e^{2it}\} = \frac{1}{5}\{\cos 2t + 2\sin 2t\} \\ &= \frac{1}{\sqrt{5}}\cos\left(2t - \text{ Arcsin}\left(\frac{2}{\sqrt{5}}\right)\right). \end{aligned}$$

c) By a change of sign in (a) we get

$$y(t) = -\frac{1}{\sqrt{2}}\cos\left(t - \frac{\pi}{4}\right).$$

d) Since $u(t) = \sin 2t = \text{ Im } e^{2it}$ and $H(2i) = \dfrac{1}{5}(1-2i)$ by (b), the real stationary solution is

$$\begin{aligned} y(t) &= \frac{1}{5}\operatorname{Im}\{(1-2i)e^{2it}\} = \frac{1}{5}\{\sin 2t - 2\cos 2t\} = \frac{1}{\sqrt{5}}\left\{\frac{1}{\sqrt{5}}\sin 2t - \frac{2}{\sqrt{5}}\cos 2t\right\} \\ &= \frac{1}{\sqrt{5}}\sin\left(2t - \text{ Arcsin}\left(\frac{2}{\sqrt{5}}\right)\right). \end{aligned}$$

Example 5.6 *Prove that the linear system*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is asymptotically stable.
Find the transfer function (the transfer matrix) for the linear system

$$(14)\quad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \qquad y(t) = x_2(t),$$

and find the real stationary response of (14) for the influence $u(t) = \cos t$.

The characteristic polynomial

$$\begin{vmatrix} -\lambda-3 & 1 \\ 4 & -\lambda-3 \end{vmatrix} = (\lambda+3)^2-4 = (\lambda+3)^2-2^2 = (\lambda+1)(\lambda+5)$$

has the two to negative roots $\lambda_1 = -1$ and $\lambda_2 = -5$. We conclude that the linear system is asymptotically stable.

The transfer function is given by

$$H(s) = \mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} + d,$$

where we in the given case have

$$\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 4 & -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{c} = (0,1), \quad d = 0,$$

thus

$$H(s) = (0,1)(s\mathbf{I}-\mathbf{A})^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since

$$s\mathbf{I}-\mathbf{A} = \begin{pmatrix} s+3 & -1 \\ -4 & s+3 \end{pmatrix}, \qquad \det(s\mathbf{I}-\mathbf{A}) = (s+1)(s+5),$$

it follows for $s \neq -1, -5$ that

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{(s+1)(s+5)}\begin{pmatrix} s+3 & 1 \\ 4 & s+3 \end{pmatrix}.$$

Then we find the transfer function

$$H(s) = \frac{1}{(s+5)(s+1)}(0,1)\begin{pmatrix} s+3 & 1 \\ 4 & s+3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{s+3}{(s+5)(s+1)} = \frac{1}{2}\left\{\frac{1}{s+1}+\frac{1}{s+5}\right\}.$$

For $u(t) = \cos t = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$ we get the real stationære response

$$\begin{aligned} y(t) &= H(i)\,\frac{1}{2}\,e^{it} + H(-i)\,\frac{1}{2}\,e^{-it} = \operatorname{Re}\{H(i)e^{it}\} = \operatorname{Re}\left\{\frac{3+i}{(5+i)(1+i)}\,e^{it}\right\} \\ &= \operatorname{Re}\left\{\frac{(3+i)(5-i)(1-i)}{(5+i)(5-i)(1+i)(1-i)}\,e^{it}\right\} = \frac{1}{26\cdot 2}\cdot\operatorname{Re}\{(16+2i)(1-i)e^{it}\} \\ &= \frac{1}{26}\operatorname{Re}\{(8+i)(1-i)e^{it}\} = \frac{1}{26}\operatorname{Re}\{(9-7i)(\cos t + i\sin t)\} = \frac{1}{26}(9\cos t + 7\sin t). \end{aligned}$$